VECTOR ANALYSIS
PROBLEM SETS

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1. Hand in: 28.09.20, count: 10Pts.

1.1. **Circle rolling on sine-graph.** A circle with radius r rolls on top of the graph of the sine function. Find a parametric representation of the path traced out by the center of the circle. Assume that the radius r is sufficiently small, such that the circle travels through the valleys of the sine graph. Hint: Any graph $y = f(x)$ can be written as a parametric curve by using x as the parameter: $\vec{r}(x) = (x, f(x))$.

1.2. **Charged particle in homogeneous magnetic field.** The trajectory of a charged particle travelling through a magnetic field $\vec{B} = (0, 0, B)$ (with B a constant), is given by

$$\vec{r}(t) = \begin{pmatrix} R\cos(\omega t) \\ R\sin(\omega t) \\ ct \end{pmatrix},$$

where R, ω and c are constants.

(i) Compute the length of the trajectory traced out between $t = 0$ and $t = 5T$, where $T = 2\pi/\omega$.

(ii) Show that the given trajectory $\vec{r}(t)$ is in agreement with Newton's law of motion $\vec{F} = m\vec{a}$, where the acting force $\vec{F}$ is given by $\vec{F} = q\vec{v} \times \vec{B}$, i.e. given by the Lorentz force.

(iii) Find an expression for ω in terms of q, B, m.

2. Hand in: 05.10.20, count: 10Pts.

2.1. **Arc length of graph.** We consider a curve given by the graph of a function $f : [a, b] \to \mathbb{R}$, $x \mapsto y = f(x)$. Use the arc length formula for curves, as given in class, to show that the arc length of the graph is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

2.2. **Gravitational force.** The gravitational potential outside earth is given in good approximation by

$$\phi(x, y, z) = -\frac{GM}{r(x, y, z)}, \qquad \text{where} \qquad r(x, y, z) = \sqrt{x^2 + y^2 + z^2}.$$

Here G is the gravitational constant, M is the mass of the earth and the coordinates have been chosen such that the center of the earth coincides with the origin. The gravitational force acting on a body of mass m is given by $\vec{F}_g = m\vec{g}$, where $\vec{g} = -\vec{\nabla}\phi$ is the gravitational field. Show that the gravitational force $\vec{F}_g$ points towards the origin and its magnitude is given by

$$\left|\vec{F}_g\right| = \frac{GMm}{r^2}.$$

3. Hand in: 12.10.20, count: 10Pts.

3.1. **Work for bringing object into orbit.** An object with mass m is brought from the surface of the earth to a height h by following the trajectory given by

$$\vec{r} : [0,1] \to \mathbb{R}^3$$
$$t \mapsto \vec{r}(t) = (r_E + th) \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \\ 0 \end{pmatrix}.$$

Here r_E is the radius of the earth.

(i) Compute the necessary work using
$$W = \int_C \vec{F}_g \cdot \vec{dr},$$
where $\vec{F}_g$ is the gravitational force as given in the previous exercise.
(ii) Compare the result with the one that would be obtained on a purely radial trajectory.
(iii) Explain why the results have to be the same.

3.2. **Path independence and closed loop.** Let $\vec{F}$ be a vectorfield defined over a domain Ω. Show that the following statementes are equivalent:

(i) $\int_C \vec{F} \cdot d\vec{r}$ is path independent,
(ii) $\int_C \vec{F} \cdot d\vec{r} = 0$ for any closed loop C.

Hint: Use $\int_C \vec{F} \cdot d\vec{r} = -\int_{C^-} \vec{F} \cdot d\vec{r}$, where C and C^- possess the same trace but opposite orientation.

4. Hand in: 2.11.20, count: 10Pts.

4.1. **Gradient fields.** For each of the following, determine for what value of the constant a the vector field $\vec{F}$ will be a gradient field, and for this value, find the corresponding potential function:

(i) $\vec{F} = \begin{pmatrix} y^2 + 2x \\ axy \end{pmatrix}$, (ii) $\vec{F} = e^{x+y} \begin{pmatrix} x + a \\ x \end{pmatrix}$.

4.2. **Divergence and curl density.** For each of the following two-dimensional vector fields, draw the vector fields in the domain $[-4,4] \times [-4,4]$ and compute the divergence and the curl density[1]:

(i) $\vec{F} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (translation)
(ii) $\vec{F} = \begin{pmatrix} y \\ 0 \end{pmatrix}$ (shear flow)
(iii) $\vec{F} = \begin{pmatrix} x \\ y \end{pmatrix}$ (uniform expansion)
(iv) $\vec{F} = \begin{pmatrix} -y \\ x \end{pmatrix}$ (solid rotation)
(v) $\vec{F} = \frac{1}{x^2+y^2} \begin{pmatrix} x \\ y \end{pmatrix}$ (radial outflow)

[1] The curl density of a vector field $\vec{F} = \begin{pmatrix} M \\ N \end{pmatrix}$ is given by $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$.

(vi) $\overrightarrow{F} = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$ (irrotational vortex)

4.3. **Greens Theorem.** Verify both forms of Greens theorem for the vector field

$$\overrightarrow{F} = \begin{pmatrix} x - y \\ x \end{pmatrix}$$

and the region bounded by the unit circle with center at the origin. Hint:

$$\int_0^{2\pi} \cos^2(x)dx = \pi.$$