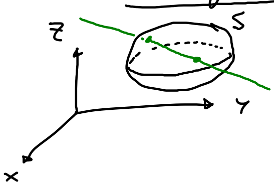
Divergence thm.

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iiint_V \vec{\nabla}\cdot\vec{F}\, dV}$$

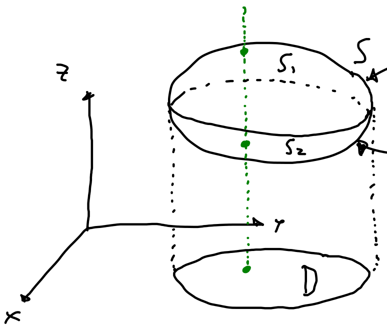
where: $\vec{\nabla}\cdot\vec{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} M \\ N \\ P \end{pmatrix} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$

is called the divergence of $\vec{F}$



Proof:

(i) S convex: **Def.:** Closed surface S is convex if any straight line meeting it does so in at most two points.



$z = f_1(x,y)$

$z = f_2(x,y)$

Let $\vec{F} = \begin{pmatrix} M \\ N \\ P \end{pmatrix}$; $\vec{F}_1 = \begin{pmatrix} M \\ 0 \\ 0 \end{pmatrix}$; $\vec{F}_2 = \begin{pmatrix} 0 \\ N \\ 0 \end{pmatrix}$; $\vec{F}_3 = \begin{pmatrix} 0 \\ 0 \\ P \end{pmatrix}$

$\longrightarrow \vec{F} = \sum_{i=1}^{3} \vec{F}_i$

$$\iiint_V \frac{\partial P}{\partial z}\, dV = \underbrace{\int_{x=\cdots}^{\cdots} \int_{y=\cdots}^{\cdots}}_{D} \int_{z=f_2(x,y)}^{f_1(x,y)} \frac{\partial P}{\partial z}(x,y,z)\, dz\, dy\, dx = \iint_D \Big(P(x,y,f_1(x,y)) - P(x,y,f_2(x,y))\Big) dy\, dx \quad (*)$$

Upper surface: $z = f_1(x,y) \longrightarrow \vec{r}(x,y) = \begin{pmatrix} x \\ y \\ f_1(x,y) \end{pmatrix} \longrightarrow \vec{r}_x \times \vec{r}_y = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f_1}{\partial x} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f_1}{\partial y} \end{pmatrix} = \begin{pmatrix} -\frac{\partial f_1}{\partial x} \\ -\frac{\partial f_1}{\partial y} \\ 1 \end{pmatrix}$

$$\longrightarrow \iint_{S_1} \vec{F}_3 \cdot d\vec{S} = \iint_D \begin{pmatrix} 0 \\ 0 \\ P \end{pmatrix} \cdot \begin{pmatrix} -\frac{\partial f_1}{\partial x} \\ -\frac{\partial f_1}{\partial y} \\ 1 \end{pmatrix} dx\,dy = \iint_D \underset{P(x,y,f_1(x,y))}{\underset{\parallel}{P}}\, dx\,dy$$

$$\longrightarrow \iint_{\underset{S_1\cup S_2}{\underset{\parallel}{S}}} \vec{F}_3 \cdot d\vec{S} = \iint_D \Big(P(x,y,f_1(x,y)) - P(x,y,f_2(x,y))\Big) dx\,dy = \iiint_V \frac{\partial P}{\partial z}\, dV = \iiint_V \vec{\nabla}\cdot\vec{F}_3\, dV$$

Similarly for $\vec{F}_1 = \begin{pmatrix} M \\ 0 \\ 0 \end{pmatrix}$; $\vec{F}_2 = \begin{pmatrix} 0 \\ N \\ 0 \end{pmatrix}$ but there, integrate w.r.t. x, y first respectively:

$$\iint_S \vec{F}_1 \cdot d\vec{S} = \iiint_V \vec{\nabla}\cdot\vec{F}_1\, dV$$

$$\iint_S \vec{F}_2 \cdot d\vec{S} = \iiint_V \vec{\nabla}\cdot\vec{F}_2\, dV$$

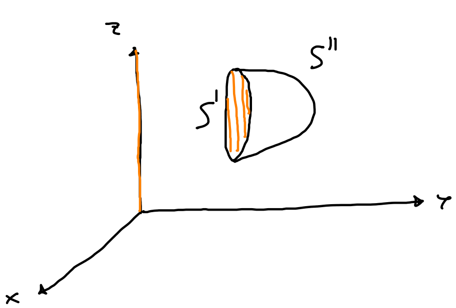
$$\longrightarrow \iint_S \vec{F}\cdot d\vec{S} = \sum_{i=1}^{3} \iint_S \vec{F}_i \cdot d\vec{S} = \sum_{i=1}^{3} \iiint_V \vec{\nabla}\cdot\vec{F}_i\, dV = \iiint_V \vec{\nabla}\cdot\vec{F}\, dV$$

$\downarrow$ div. thm. for S convex.

(ii) S semi-convex **Def.:** S is called semi-convex, if x-y-z-axes can be chosen, s.t. any line parallel to one of the axes and meeting S either:

(i) does so in at most two points or

(ii) has a portion of finite length in common with S.



Argument as above: $\iint_{S''} \vec{F}_3 \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{F}_3 \, dV$

But: $\iint_{S'} \vec{F}_3 \cdot d\vec{S} = 0$ (because: $\vec{F}_3 \cdot \underbrace{(\vec{r}_u \times \vec{r}_v)}_{\text{perpendicular to}} = 0$)

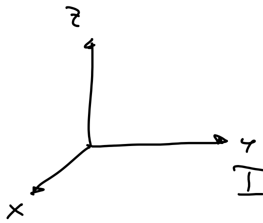
$\longrightarrow \iint_S \vec{F}_3 \cdot d\vec{S} = \underbrace{\iint_{S'} \cdots}_{=0} + \iint_{S''} \cdots = \iiint_V \vec{\nabla} \cdot \vec{F}_3 \, dV$

Other directions need no modification.

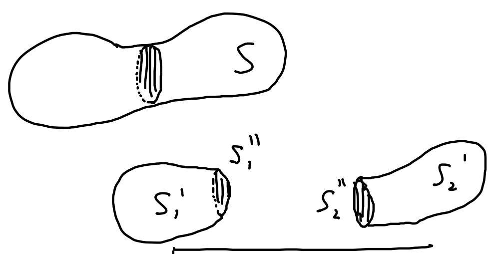
$\longrightarrow \iint_S \vec{F} \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{F} \, dV$

$\downarrow$ S semi-convex.

(iii)



Idea:



not convex or semi-convex?

both semi-convex!

$S_1 = S_1' \cup S_1''$

$S_2 = S_2' \cup S_2''$

$S = S_1' \cup S_2'$

We have: $\boxed{\iiint_V \vec{\nabla} \cdot \vec{F} \, dV} = \iiint_{V_1} \vec{\nabla} \cdot \vec{F} \, dV + \iiint_{V_2} \vec{\nabla} \cdot \vec{F} \, dV$

$= \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}$

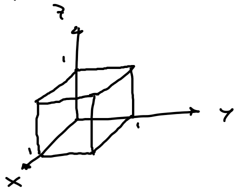
$= \iint_{S_1'} \vec{F} \cdot d\vec{S} + \iint_{S_1''} \vec{F} \cdot d\vec{S} + \iint_{S_2'} \vec{F} \cdot d\vec{S} + \iint_{S_2''} \vec{F} \cdot d\vec{S}$

$= \iint_{S_1'} \vec{F} \cdot d\vec{S} + \iint_{S_2'} \vec{F} \cdot d\vec{S} = \boxed{\iint_S \vec{F} \cdot d\vec{S}}$

$\iint_{S_1''} \vec{F} \cdot d\vec{S} = - \iint_{S_2''} \vec{F} \cdot d\vec{S}$ $\qquad\qquad$ $S = S_1' \cup S_2'$

$\square$

Ex: Flux of $\vec{F} = \begin{pmatrix} xy \\ yz \\ xz \end{pmatrix}$ through surface of cube: $[0,1]^3$



div. thm.

$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{F} \, dV = \iiint_V (x+y+z) \, dV$

$\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xz) = y + z + x$

$= \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} (x+y+z) \, dx \, dy \, dz = \dots = \underline{\frac{3}{2}}$

Ex: Gauss's law [did this in 2D on Oct. 20]

$\boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}}$ (one of Maxwell's eqn.)

$$\hookrightarrow \iiint_V \vec{\nabla}\cdot\vec{E}\, dV = \iiint_V \frac{\rho}{\varepsilon_0}\, dV = \frac{1}{\varepsilon_0}\underbrace{Q}_{\text{total charge in } V}$$

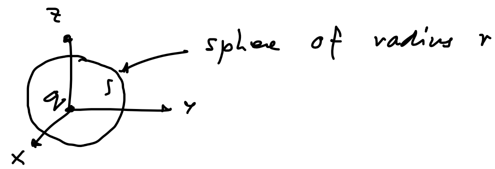
// div. thm.

$$\iint_S \vec{E}\cdot d\vec{s}$$

↖ Flux of electric $\vec{E}$ field through surface S

Gauss's law !

Ex : Point charge q at origin :



$$\longrightarrow \underline{\frac{q}{\varepsilon_0}} = \iiint_V \frac{\rho}{\varepsilon_0}\, dV \overset{\text{Gauss}}{=} \iiint_V \vec{\nabla}\cdot\vec{E}\, dV \underset{\text{div. thm.}}{=} \iint_S \vec{E}\cdot d\vec{s} = \iint_S |\vec{E}|\, \vec{N}\cdot d\vec{s}$$

$\vec{E}$ has to point radially (by symmetry)

$$= |\vec{E}| \underbrace{\iint_S \vec{N}\cdot d\vec{s}}_{=4\pi r^2} = \underline{4\pi r^2 |\vec{E}|} \longrightarrow |\vec{E}| = \frac{q}{4\pi\varepsilon_0 r^2}$$

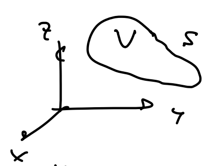
$\vec{E}$ radially $\longrightarrow$ $\underline{\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}}$; $r = \sqrt{x^2+y^2+z^2}$, $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Force on charge Q : $\vec{F} = Q\vec{E} \longrightarrow$

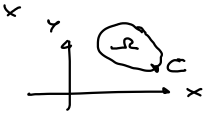
$$\boxed{\vec{F} = \frac{qQ}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3} \qquad |\vec{F}| = \frac{qQ}{4\pi\varepsilon_0 r^2} \qquad \text{Coulomb's law !}}$$

Rem.: Div. thm. works in any dimension !

n = 3: $\displaystyle\iiint_V \vec{\nabla}\cdot\vec{F}\, dV = \iint_S \vec{F}\cdot d\vec{s}$ (Gauss)



n = 2: $\displaystyle\iint_\Omega \underbrace{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)}_{\vec{\nabla}\cdot\vec{F} \text{ in 2D}} dV = \int_C \vec{F}\cdot d\vec{N}$ (Green)



n = 1: $\displaystyle\int_a^b \frac{df}{dx}\, dx = f(b) - f(a)$ (Fund. thm. of calculus)