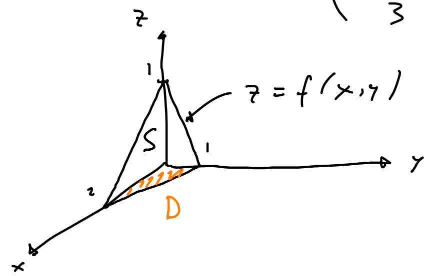
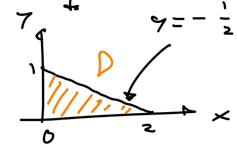
$\boxed{7.2}$ (i) $\vec{F} = \begin{pmatrix} 6z \\ -3y \\ 3 \end{pmatrix}$; Plane: $x + 2y + 2z = 2$ in 1st Octant $(x,y,z \geq 0)$



$\longrightarrow \vec{r} : D \to \mathbb{R}^3$

$(x,y) \longmapsto \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 - \frac{1}{2}x - y \end{pmatrix}$



$y = -\frac{1}{2}x + 1$

$z = 1 - \frac{1}{2}x - y$

$$\longrightarrow \underline{\text{Flux}} = \iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \underbrace{(\vec{r}_x \times \vec{r}_y)}_{\begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}} \, dx\,dy = \underline{\int_{x=0}^{2} \int_{y=0}^{-\frac{1}{2}x+1} \begin{pmatrix} 6z \\ -3y \\ 3 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix} dy\,dx}$$

(ii) $\vec{r}(u,v) = \begin{pmatrix} 5\cos(v) \\ 5\sin(v) \\ u \end{pmatrix}$; $(u,v) \in [0,10] \times [0,2\pi]$

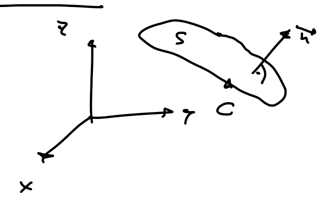
$\vec{F} = \begin{pmatrix} y \\ x \\ z^2 \end{pmatrix}$; $\vec{F}(\vec{r}(u,v)) = \begin{pmatrix} 5\sin(v) \\ 5\cos(v) \\ u^2 \end{pmatrix}$; $\vec{r}_u \times \vec{r}_v = \begin{pmatrix} -5\cos(v) \\ -5\sin(v) \\ 0 \end{pmatrix}$

$$\longrightarrow \underline{\text{Flux}} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, du\,dv$$

$$= \underline{\int_{u=0}^{10} \int_{v=0}^{2\pi} -50 \sin(v)\cos(v)\, du\,dv}$$

<u>Recap:</u> <u>Stokes Thm.</u>

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}}$$



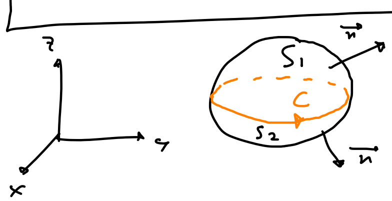
<u>Applications:</u> Maxwell's equns.:

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \longrightarrow$ <u>Faraday's</u> law of induction

$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \longrightarrow$ <u>Ampere's</u> law

<u>Thm.:</u> $\boxed{\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = 0 \quad \text{for any closed surface } S}$

<u>Proof:</u>



Idea: Pick <u>closed curve C</u> which lies in S.

Then: $S = S_1 \cup S_2$

We have (by Stokes):

$$\int_C \vec{F} \cdot d\vec{r} = \iint_{S_1} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

$$\int_C \vec{F} \cdot d\vec{r} = -\iint_{S_2} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

due to orientation of S_2 and C.

$$\longrightarrow \iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \iint_{S_1} (\vec{\nabla}\times\vec{F})\cdot d\vec{S} + \iint_{S_2} (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$$

$$= \int_C \vec{F}\cdot d\vec{r} - \int_C \vec{F}\cdot d\vec{r} = 0 \qquad \square$$

'Rough' interpretation: Flux of $\vec{G} = \vec{\nabla}\times\vec{F}$ through closed surface S is zero!
I.e. 'what flows in flows out', Like for conserved quantity.
With interpretation of <u>divergence</u> as source strength, we expect:

$$\vec{\nabla}\cdot\vec{G} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0.$$

this is true! (see ex.).

Application to gradient fields:

<u>Thm.</u> Let $\vec{F}$ be defined in a simply connected region.
The following are equivalent:

(i) $\vec{F} = \vec{\nabla}\phi$ for some potential ϕ

(ii) $\vec{\nabla}\times\vec{F} = \vec{0}$

(have seen this in 2D on Oct. 5)

<u>Proof:</u> <u>(i) $\Rightarrow$ (ii):</u> Computation: $\vec{\nabla}\times\vec{F} = \vec{\nabla}\times\vec{\nabla}\phi = \vec{0}$
(see ex.)

<u>(ii) $\Rightarrow$ (i):</u> $\vec{\nabla}\times\vec{F} = \vec{0} \longrightarrow \int_C \vec{F}\cdot d\vec{r} = \iint_S \underbrace{(\vec{\nabla}\times\vec{F})}_{=\vec{0}}\cdot d\vec{S} = 0$

closed loop

surface whose boundary is C.

$\xrightarrow{}$ there exists potential ϕ, s.t. $\vec{F} = \vec{\nabla}\phi$.

fund. thm. of line int.

$\square$

<u>Ex:</u> <u>Electrostatics:</u> Maxwell's eqns:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \text{(Gauss)} \qquad (1)$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \text{(Faraday)} \qquad (2)$$

If static: $\frac{\partial\vec{B}}{\partial t} = \vec{0}$

(2) $\longrightarrow \vec{\nabla}\times\vec{E} = \vec{0} \longrightarrow \underline{\vec{E} = \vec{\nabla}\phi}$

this in (1) $\longrightarrow \vec{\nabla}\cdot\vec{\nabla}\phi = \frac{\rho}{\varepsilon_0}$

I.e. $\quad \Delta \phi = \frac{\rho}{\varepsilon_0}$ (Poisson's eqn.).

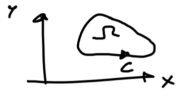
(Ampère: $\quad \nabla \times \vec{B} = \mu_0 \left(\vec{J} + \cancel{\frac{\partial \vec{E}}{\partial t}} \right)$ $\quad$ Magnetostatics

$\nabla \cdot \vec{B} = 0 \longrightarrow \vec{B} = \nabla \times \vec{A}$ $\quad \boxed{\vec{A}, \phi}$)

Divergence thm.

Overview:

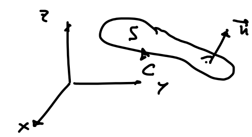
Green, tang.



$\vec{F} = \begin{pmatrix} M \\ N \end{pmatrix}$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA \longrightarrow$$

Stokes:



$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

Green, normal:

$$\int_C \vec{F} \cdot d\vec{N} = \iint_\Omega \underbrace{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right)}_{= \nabla \cdot \vec{F}} dA \longrightarrow$$

Divergence thm. (Gauss)

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V \nabla \cdot \vec{F} \, dV$$