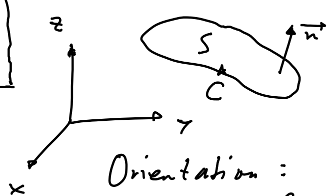
Recap: **Stokes Thm.**

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S}}$$



Orientation: Walk along C on upper side, interior to the left.

Details:

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\underset{\text{curve}}{\vec{r}(t)})\cdot\frac{d\vec{r}}{dt}(t)\,dt$$

Line integral of vector field $\vec{F}$, called circulation of $\vec{F}$ around C.

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \iint_D (\vec{\nabla}\times\vec{F})(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)(u,v)\,du\,dv$$

surface integral of $\vec{\nabla}\times\vec{F}$ i.e. flux of $\vec{\nabla}\times\vec{F}$ throug S.

Curl:

$$\vec{\nabla}\times\vec{F} = \begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}\times\begin{pmatrix}M\\ N\\ P\end{pmatrix} = \begin{pmatrix}\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z}\\ \frac{\partial M}{\partial z}-\frac{\partial P}{\partial x}\\ \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\end{pmatrix}$$

direction of $\vec{\nabla}\times\vec{F}$: axis of rotation

magn. of $\vec{\nabla}\times\vec{F}$: how much it rotates

Idea of proof: Take problem into x-y plane, use Green.

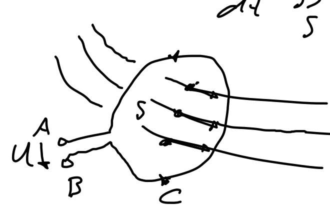
Application: **Faraday's law:** $\underline{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$ (Maxwell's eqn.)

$$\iint_S (\vec{\nabla}\times\vec{E})\cdot d\vec{S} = \iint_S -\frac{\partial\vec{B}}{\partial t}\cdot d\vec{S}$$

$$= -\frac{d}{dt}\underbrace{\iint_S \vec{B}\cdot d\vec{S}}_{=\Phi\text{ : Magn. flux}} = -\frac{d\Phi}{dt}$$

$\|$ Stokes

$$\int_C \vec{E}\cdot d\vec{r} = U\text{: induced voltage}$$



Other Application: Ampere's law

$\underbrace{}_{}$ not due to Ampere

$$\vec\nabla \times \vec B = \mu_0 \left(\vec J + \varepsilon_0 \frac{\partial \vec E}{\partial t}\right) \quad \text{(Maxwell eqn.)}$$

$\vec J$: current density

Assumption: time indep. $\longrightarrow \frac{\partial \vec E}{\partial t} = \vec 0$

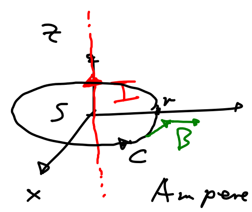
$$\longrightarrow \underline{\vec\nabla \times \vec B = \mu_0 \vec J}$$

$$\iint_S (\vec\nabla \times \vec B)\cdot d\vec s = \underbrace{\iint_S \mu_0 \vec J \cdot d\vec s}_{\text{current through surface } S} = \mu_0 I$$

// Stokes

$$\underbrace{\int_C \vec B \cdot d\vec s}_{\text{circulation of } \vec B \text{ around } C}$$

Ex: Wire along z-axis



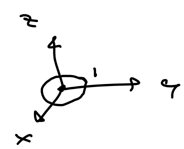
Ass: $\vec B$ - field points in circumferential direction

then: $\int_C \vec B \cdot d\vec r = |\vec B|\, 2\pi r \overset{\text{Ampere}}{=} \mu_0 I$

$$\longrightarrow \underline{|\vec B| = \frac{\mu_0 I}{2\pi r}}$$

Computational Ex.

$\vec F = \begin{pmatrix} z \\ x \\ y \end{pmatrix}$; We compute work around unit circle in x-y-plane at origin.



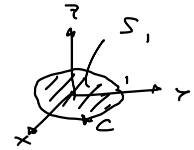
(i) direct: $W = \int_C \vec F \cdot d\vec r = \int_a^b \vec F(\vec r(t)) \cdot \frac{d\vec r}{dt}(t)\, dt$

$$= \int_0^{2\pi} \begin{pmatrix} 0 \\ \cos(t) \\ \sin(t) \end{pmatrix} \cdot \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{pmatrix} dt$$

$\vec r(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \end{pmatrix}$

$$= \int_0^{2\pi} \cos^2(t)\, dt = \underline{\pi}$$

(ii) Stokes on unit disk:



$$W = \int_C \vec F \cdot d\vec r = \iint_{S_1} (\vec\nabla \times \vec F)\cdot d\vec s = \int_0^{2\pi}\int_0^1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} du\, dv = \int_0^{2\pi}\int_0^1 u\, du\, dv$$

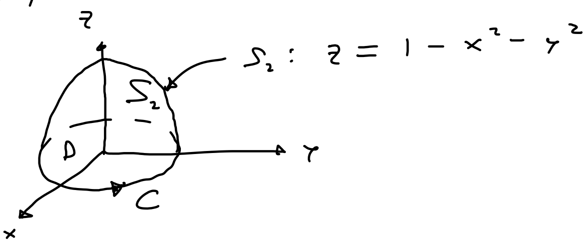
$$= \underline{\pi}$$

$\vec r(u,v) = \begin{pmatrix} u\cos(v) \\ u\sin(v) \\ 0 \end{pmatrix}$; $(u,v) \in [0,1] \times [0, 2\pi]$

$$\underbrace{\vec{r}_u}_{\frac{\partial \vec r}{\partial u}} \times \vec{r}_v = \begin{pmatrix} \cos(v) \\ \sin(v) \\ 0 \end{pmatrix} \times \begin{pmatrix} -u\sin(v) \\ u\cos(v) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}$$

$$\vec{\nabla} \times \vec{F} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} z \\ x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(iii) Stokes on paraboloid



$$\vec{r}(x,y) = \begin{pmatrix} x \\ y \\ 1 - x^2 - y^2 \end{pmatrix}$$

$$\vec{r}_x \times \vec{r}_y = \begin{pmatrix} 1 \\ 0 \\ -2x \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -2y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 1 \end{pmatrix}$$

$$\longrightarrow \quad W = \int_C \vec{F} \cdot d\vec{r} = \iint_{S_2} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \iint_D \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2x \\ 2y \\ 1 \end{pmatrix} dx\,dy$$

$$= \iint_D (2x + 2y + 1)\, dx\,dy$$

$$= \int_{\varphi=0}^{2\pi} \int_{r=0}^{1} \big(2r\cos(\varphi) + 2r\sin(\varphi) + 1\big)\, r\, dr\, d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \Big(\frac{2}{3} \underbrace{\big(\cos(\varphi) + \sin(\varphi)\big)}_{\longrightarrow\, 0} + \frac{1}{2} \Big) d\varphi$$

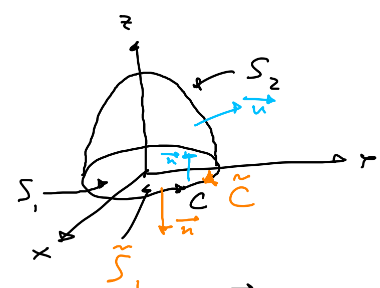
$$= \underline{\pi}$$

The previous computation tells us:

$$\iint_{S_1} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \iint_{S_2} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

I.e. with $\vec{G} = \vec{\nabla} \times \vec{F}$, then:

$$\iint_{S_1} \vec{G} \cdot d\vec{S} = \iint_{S_2} \vec{G} \cdot d\vec{S}$$



We have:
$$\iint_{\tilde{S}_1} \vec{G} \cdot d\vec{S} = -\iint_{S_1} \vec{G} \cdot d\vec{S} = -\iint_{S_2} \vec{G} \cdot d\vec{S}$$

$$\longrightarrow \quad \boxed{\iint_S \vec{G} \cdot d\vec{S} = \iint_{S_2} \vec{G} \cdot d\vec{S} + \iint_{\tilde{S}_1} \vec{G} \cdot d\vec{S} = 0}$$

I.e.
$$\boxed{\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = 0 \quad \text{for any closed surface}}$$