Curl of a vector field

We saw: curl density of 2D-vector field:

$$\vec{F} = \begin{pmatrix} M \\ N \end{pmatrix}$$

$$\longrightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$: measures how much vector field rotates (counter clock wise)

In 3D: Let: $\vec{F} = \begin{pmatrix} M \\ N \\ 0 \end{pmatrix}$ (no z-component) & $M = M(x,y)$, $N(x,y)$

$\longrightarrow$ Amount of how this vector field rotates is $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

Axis of rotation is the z-axis.

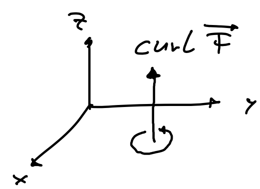
Written as a vector: $\operatorname{curl} \vec{F} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \end{pmatrix}$

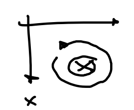
Ex: solid rotation in 3D:

$$\vec{F} = \frac{1}{x^2+y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \longrightarrow \operatorname{curl} \vec{F} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

magnitude: how much it rotates

direction: axis of rotation (if you look down on the x-y-plane it rotates counter clock wise if $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} > 0$)





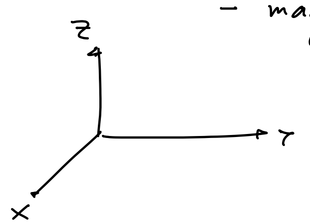
Generalization to arbitrary vector fields: $\vec{F} = \begin{pmatrix} M \\ N \\ P \end{pmatrix}$:

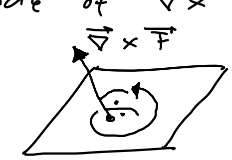
Def:
$$\operatorname{curl} \vec{F} = \begin{pmatrix} \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \\ \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \\ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \end{pmatrix} ; \quad \text{with } \vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$\longrightarrow \underline{\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F}}$$

interpretation stays the same:

- direction of $\vec{\nabla} \times \vec{F}$: axis of rotation
- magnitude of $\vec{\nabla} \times \vec{F}$: how much it rotates

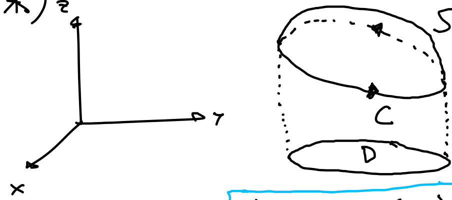




$$\left(\begin{array}{l} \vec{\nabla}\phi \; ; \; \Delta = \vec{\nabla} \cdot \vec{\nabla}\phi \\ \vec{\nabla} \cdot \vec{F} \\ \vec{\nabla} \times \vec{F} \end{array} \right)$$

Stokes Thm.:

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot d\vec{S}} \quad (*)$$



Proof: Restrict to case where S is a graph:

Surface S: $\vec{r}(x,y) = \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$

$\longrightarrow$ $\vec{r}_x = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x} \end{pmatrix}$; $\vec{r}_y = \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y} \end{pmatrix} \longrightarrow \vec{r}_x \times \vec{r}_y = \boxed{\begin{pmatrix} -\frac{\partial f}{\partial x} \\ -\frac{\partial f}{\partial y} \\ 1 \end{pmatrix}}$; $\nabla\times\vec{F} = \boxed{\begin{pmatrix} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{pmatrix}}$, $\vec{F} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}$

$\longrightarrow$ r.h.s. of $(*)$:

$$\iint_S (\nabla\times\vec{F})\cdot d\vec{S} = \iint_D \boxed{(\nabla\times\vec{F})}\cdot\boxed{(\vec{r}_x\times\vec{r}_y)}\,dx\,dy$$

$$= \iint_D \left(\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\left(-\frac{\partial f}{\partial x}\right) + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\left(-\frac{\partial f}{\partial y}\right) + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right) dx\,dy$$

Curve C: $\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$; $t\in[a,b]$

$\longrightarrow$ l.h.s. of $(*)$:

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \begin{pmatrix} P(\vec{r}(t)) \\ Q(\dots) \\ R(\dots) \end{pmatrix} \cdot \overbrace{\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix}}^{\frac{d\vec{r}}{dt}(t)} dt$$

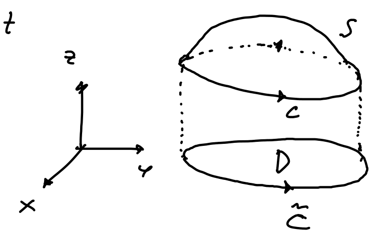
$$= \int_a^b \left(P(\vec{r}(t))\dot{x}(t) + Q(\dots)\dot{y}(t) + R(\dots)\dot{z}(t) \right) dt$$

$$= \int_a^b \left(\left(P + R\frac{\partial f}{\partial x}\right)\dot{x} + \left(Q + R\frac{\partial f}{\partial y}\right)\dot{y} \right) dt$$

$$\dot{z}(t) = \frac{dz}{dt}(t) = \frac{\partial f}{\partial x}(x(t),y(t))\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(\dots)\frac{dy}{dt}(t)$$

$$z(t) = f(x(t),y(t)) \quad (\text{Graph!})$$

$$= \int_a^b \begin{pmatrix} P + R\frac{\partial f}{\partial x} \\ Q + R\frac{\partial f}{\partial y} \end{pmatrix}\cdot\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} dt$$



$$= \int_{\tilde{C}} \begin{pmatrix} P + R\frac{\partial f}{\partial x} \\ Q + R\frac{\partial f}{\partial y} \end{pmatrix}\cdot d\tilde{\vec{r}}$$

$$= \iint_D \left(\frac{\partial}{\partial x}\left(Q + R\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial f}{\partial x}\right) \right) dx\,dy$$

$\boxed{\text{Green:}}$ $\int_{\tilde{C}} \begin{pmatrix} M \\ N \end{pmatrix}\cdot d\tilde{\vec{r}} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$Q(x,y,f(x,y))$

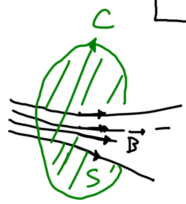
$\downarrow$

$$= \iint_D \Bigg(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial f}{\partial x} + \left(\frac{\partial R}{\partial x} + \cancel{\frac{\partial R}{\partial z}\frac{\partial f}{\partial x}}\right)\frac{\partial f}{\partial y} + R\cancel{\frac{\partial^2 f}{\partial x\partial y}}$$
$$- \frac{\partial P}{\partial y} - \frac{\partial P}{\partial z}\frac{\partial f}{\partial y} - \left(\frac{\partial R}{\partial y} + \cancel{\frac{\partial R}{\partial z}\frac{\partial f}{\partial y}}\right)\frac{\partial f}{\partial x} - R\cancel{\frac{\partial^2 f}{\partial y\partial x}} \Bigg) dx\,dy$$

$$= \iint_D (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} \qquad \square$$

Application: <u>Faraday's law of induction</u>

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}} \qquad \text{(One of Maxwell's eqn.)}$$



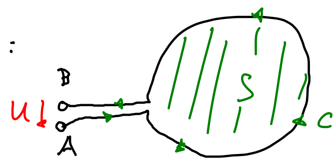
integrating this over surface S yields:

$$\iint_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{S} = \iint_S \left(-\frac{\partial \vec{B}}{\partial t}\right) \cdot d\vec{S}$$

$$\overset{\text{Stokes}}{=} \; \int_C \vec{E} \cdot d\vec{r} \qquad\qquad = -\frac{d}{dt} \underbrace{\iint_S \vec{B} \cdot d\vec{S}}_{=: \Phi(t)}$$

$=: \Phi(t)$: <u>magnetic flux</u> through S.

If C is made into:



then, since $\vec{E} = \frac{\vec{F}}{q} \;\frac{\text{(force)}}{\text{(charge)}} \longrightarrow \int_C \vec{E} \cdot d\vec{r}$ is work per charge to bring charge from A to B.

since voltage $= \frac{\text{work per}}{\text{charge}} \longrightarrow \int_C \vec{E} \cdot d\vec{r}$ is the <u>voltage difference</u> from A to B: U

I.e. Faraday's law is:

$$\boxed{U = \int_C \vec{E} \cdot d\vec{r} = -\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{S} = -\frac{d\Phi}{dt}}$$

I.e. induced voltage in loop is equal to the change in magnetic flux! $\quad |||$