Heat eqn.

therm. energy in $\Omega = \iint_\Omega \rho c u \, dA$ ← temp.

heat flux $= -k \vec{\nabla} u$ (Fourier)

change of therm. energy $=$ $-$ total outflux

$$\frac{d}{dt}\left(\iint_\Omega \rho c \underbrace{u}_{u(t,x,y)} \, dA\right) = -\int_C (-k\vec{\nabla} u)\cdot d\vec{N}$$

$$\| \qquad\qquad \underset{\text{Green}}{=} \iint_\Omega \vec{\nabla}\cdot(k\vec{\nabla} u)\, dA$$

$$\iint_\Omega \rho c \frac{\partial u}{\partial t}\, dA$$

$$\left(\vec{\nabla}\cdot\vec{B} = \frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} \qquad \vec{B} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}\right)$$

$$\longrightarrow \iint_\Omega \left(\rho c \frac{\partial u}{\partial t} - k\vec{\nabla}\cdot(\vec{\nabla} u)\right) dA = 0$$

$$\xrightarrow{\Omega \text{ arb.}} \underline{\frac{\partial u}{\partial t} = \frac{k}{\rho c}\vec{\nabla}\cdot(\vec{\nabla} u)} \qquad \vec{\nabla} u = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix}$$

Using:

$$\vec{\nabla}\cdot(\vec{\nabla} u) = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}\cdot\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Delta u$$

$$\left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

$$\longrightarrow \boxed{\frac{\partial u}{\partial t} = \frac{k}{\rho c}\Delta u}$$ Heat eqn.

Static case: no time dependence $\longrightarrow \boxed{\Delta u = 0}$ Laplace eqn.

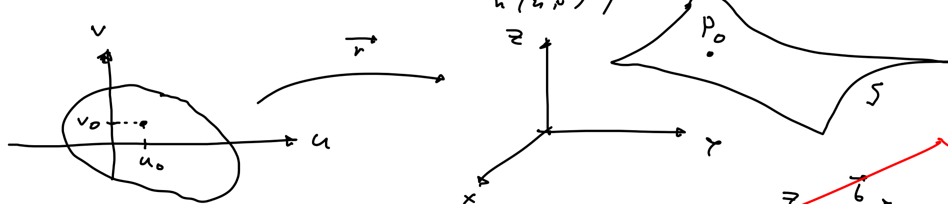
$$\left[\vec{\nabla}^2 = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}\cdot\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \ldots\right]$$

Recap:

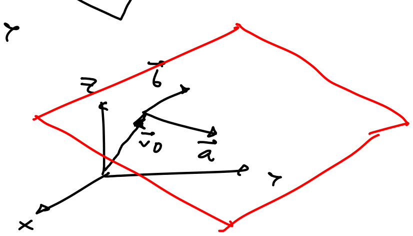
Surface: $\vec{r}: D \longrightarrow \mathbb{R}^3$

$$(u,v) \longmapsto \vec{r}(u,v) = \begin{pmatrix} f(u,v) \\ g(u,v) \\ h(u,v) \end{pmatrix}$$

$P_0 = \vec{r}(u_0, v_0)$



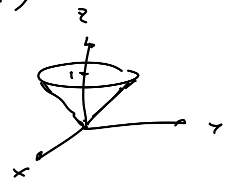
Ex: $\vec{r}(u,v) = \vec{v_0} + u\vec{a} + v\vec{b}$: plane



Ex: Surface of revolution

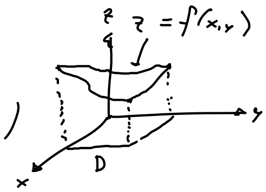
$$\vec{r} : [0,1] \longrightarrow \mathbb{R}^3$$
$$u \longmapsto \vec{r}(u) = \begin{pmatrix} u \\ 0 \\ u \end{pmatrix}$$

Then:
$$\vec{r}(u,v) = \begin{pmatrix} \cos(v) & -\sin(v) & 0 \\ \sin(v) & \cos(v) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ 0 \\ u \end{pmatrix}$$
$$= \begin{pmatrix} u\cos(v) \\ u\sin(v) \\ u \end{pmatrix} \quad : \quad \underline{\text{cone}}$$



Graphs: Let $f(x,y)$ be given.

$\longrightarrow$ The Graph $z = f(x,y)$ describes the surface: $(x,y) \longmapsto (x, y, f(x,y))$



Surface area: Let $\vec{r}(u,v)$ be given.

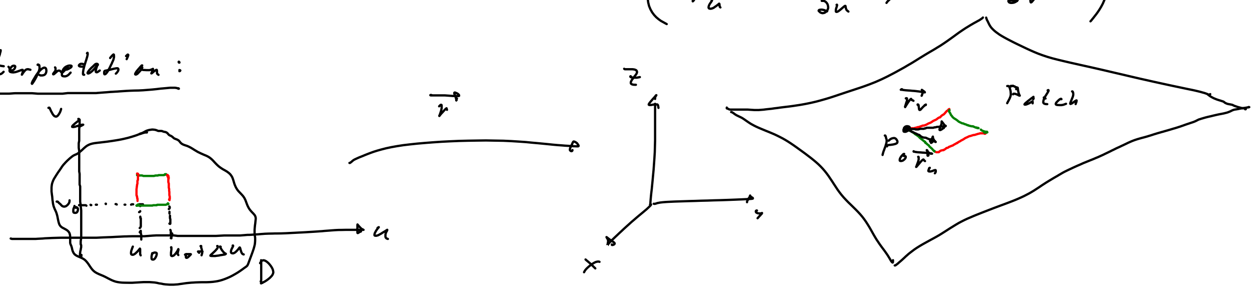
$$\vec{r} : D \longrightarrow \mathbb{R}^3$$
$$(u,v) \longmapsto \vec{r}(u,v)$$

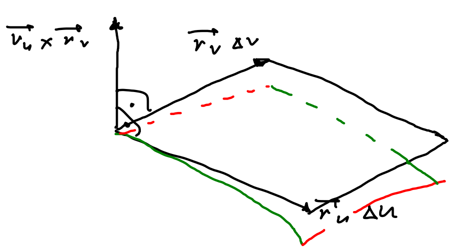
$$\boxed{A = \iint_D |\vec{r}_u \times \vec{r}_v| \, du\, dv}$$

integral over parameter domain D

$$\left(\vec{r}_u = \frac{\partial \vec{r}}{\partial u} \; ; \; \vec{r}_v = \frac{\partial \vec{r}}{\partial v} \right)$$

Interpretation:





$$\Delta A \cong |\vec{r}_u \Delta u \times \vec{r}_v \Delta v|$$
$$= |\vec{r}_u \times \vec{r}_v| \, \Delta u \, \Delta v$$
$$A \cong \sum \Delta A$$

$$\longrightarrow \underline{A = \iint_D |\vec{r}_u \times \vec{r}_v| \, du\, dv}$$

Surface integrals

(i) of scalar functions

Def: Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a scalar field. Then the surface integral of f over S is:

$$\iint_S f\, dS = \iint_D f(\vec{r}(u,v)) |\vec{r}_u \times \vec{r}_v| \, du\, dv$$

Ex: $f = \rho$ with ρ defined only on the surface S and: $\rho(x,y,z)$: mass per unit area.

$$\longrightarrow \iint_S \rho \, dS = \iint_D \rho(\vec{r}(u,v)) \, |\vec{r}_u \times \vec{r}_v| \, du\,dv = \text{Mass}.$$

(ii) <u>of vector fields</u>:

Def: Let $\vec{F}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a vector field. Then the surface integral of $\vec{F}$ over S is:

$$\iint_S \vec{F} \cdot d\vec{S} = \underbrace{\iint_S \vec{F} \cdot \vec{N} \, dS}_{\text{surf. int. of the scalar field } \vec{F}\cdot\vec{N}},$$

where $\vec{N}$ is the unit normal vector to S.

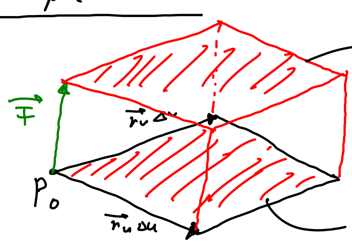
We have:

$$\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \longrightarrow \underline{\iint_S \vec{F} \cdot d\vec{S}} = \iint_S \vec{F} \cdot \vec{N} \, dS$$

$$= \iint_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \, |\vec{r}_u \times \vec{r}_v| \, du\,dv$$

$$= \underline{\iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \, du\,dv}$$

<u>Interpretation</u>:



fluid at $t = 1s$

} volume of fluid passing through surface patch in $1s$.

fluid at $t = 0s$

Let $\vec{F}$ be velocity field $|\vec{F}| = 17$

i.e. at P_0, the fluid speed is $17\,m/s$

$\longrightarrow$ Fluid passing through patch in $1s$:

$$\Delta \text{Flux} = \vec{F} \cdot (\vec{r}_u \Delta u \times \vec{r}_v \Delta v)$$

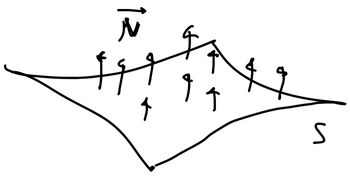
$$= \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \, \Delta u \, \Delta v$$

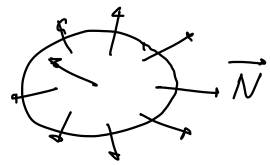
$$\longrightarrow \text{total flux: } \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \, \Delta u \, \Delta v$$

I.e. $\boxed{\iint_S \vec{F} \cdot d\vec{S} = \text{Flux}}$

Remark: Orientation

We call a surface S *orientable* (or two-sided) if it is possible to define a field $\vec{N}$ of unit normal vectors that varies continuously with position.

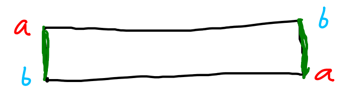




Convention: For closed surface (e.g. sphere) we take $\vec{N}$ to point outward.

Ex: Non orientable surface:

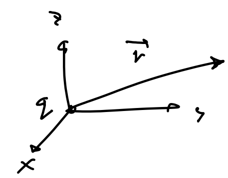
Möbius strip: piece of paper:



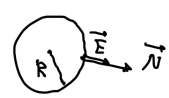
glue green on green with corners a–a, b–b together.

Ex: Flux of electric field of point charge q through sphere of radius R:

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{|\vec{r}|^3} \qquad \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



$$\underline{\underline{\text{Flux}}} = \iint_S \vec{E} \cdot d\vec{S}$$



$$= \iint_S \vec{E} \cdot \vec{N}\, dS$$

$$= \iint_S |\vec{E}|\, dS$$

$$= \iint_S \frac{q}{4\pi\varepsilon_0 R^2}\, dS = \frac{q}{4\pi\varepsilon_0 R^2} \underbrace{\iint_S dS}_{= 4\pi R^2} = \underline{\underline{\frac{q}{\varepsilon_0}}}$$