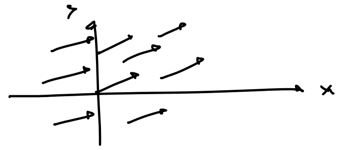
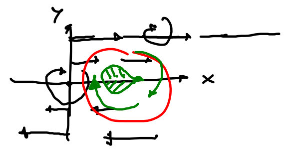
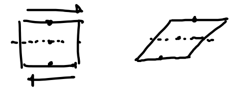
Remark on $\boxed{4.2}$:

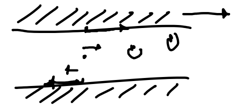
(i) $\vec{F} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



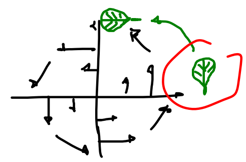
(ii) $\vec{F} = \begin{pmatrix} y \\ 0 \end{pmatrix}$



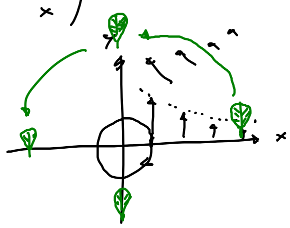




(iv) $\vec{F} = \begin{pmatrix} -y \\ x \end{pmatrix}$



(vi) $\vec{F} = \frac{1}{x^2+y^2} \begin{pmatrix} -y \\ x \end{pmatrix}$



$$\vec{r}(t) = \begin{pmatrix} r\cos(\omega t) \\ r\sin(\omega t) \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} -r\omega \sin \\ r\omega \cos \end{pmatrix} = \omega \begin{pmatrix} -y \\ x \end{pmatrix}$$

$x = r\cos$

$y = r\sin$

$\longrightarrow$ curl dens.$(\vec{v}) = 2\omega$

Stokes thm.: Overview:

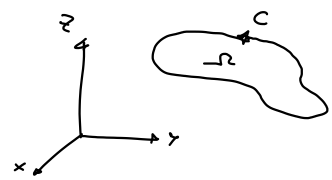
Have seen: Green's thm.: (tang. form)

$\vec{F} = \begin{pmatrix} M \\ N \end{pmatrix}$.

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA}$$

Is a thm. in x-y-plane

Goal: Version for Ω a surface in $\mathbb{R}^3$:



$\longrightarrow$ Stokes thm.

$$\underbrace{\int_C \vec{F} \cdot d\vec{r}}_{\text{line integral}} = \underbrace{\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{s}}_{\text{surface integral of vector field } \vec{\nabla} \times \vec{F}}$$

$\longrightarrow$ need to understand

- $\hookrightarrow$ surfaces
- $\hookrightarrow$ surface integrals
- $\hookrightarrow$ Operator $\vec{\nabla} \times \vec{F}$ (curl)

Surfaces

Def. A parametric representation of a surface is a function:

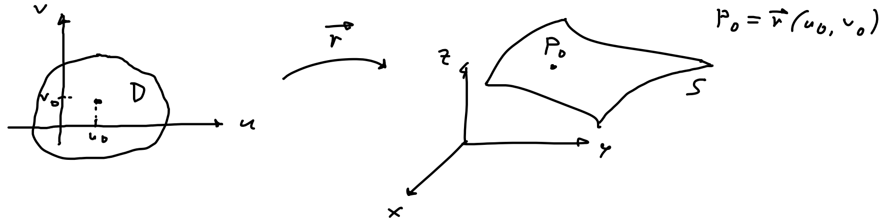
$$\vec{r}: D \longrightarrow \mathbb{R}^3$$

$$(u,v) \longmapsto \vec{r}(u,v) = \begin{pmatrix} f(u,v) \\ g(u,v) \\ h(u,v) \end{pmatrix},$$

where f, g, h are real valued functions, $D \subset \mathbb{R}^2$.

The set: $S = \{ \vec{r}(u,v) : (u,v) \in D \}$

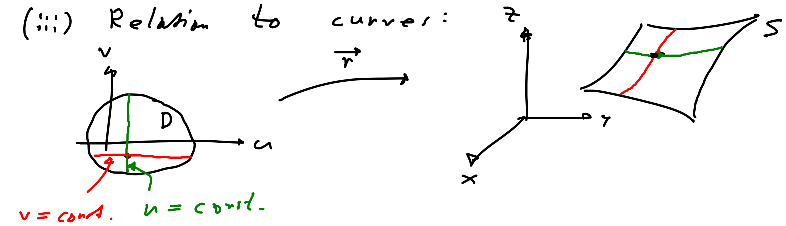
is called the surface.



Remarks: (i) u, v : parameters

(ii) D : parameter domain

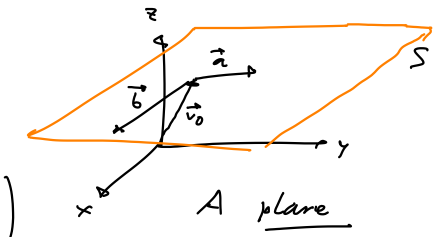
(iii) Relation to curves:



I.e. it is 2-D extension of curve.

Ex: (i) $\vec{r} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$(u,v) \longmapsto \vec{r}(u,v) = \underbrace{\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}}_{=\vec{v}_0} + u \underbrace{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}}_{\vec{a}} + v \underbrace{\begin{pmatrix} 17 \\ \sqrt{2} \\ -\pi \end{pmatrix}}_{\vec{b}}$$

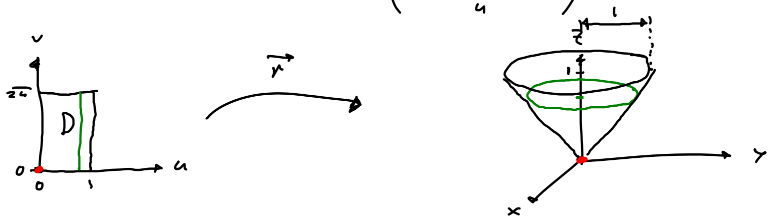


A plane

$\left(\text{In general: } \vec{r}(u,v) = \vec{v}_0 + u\vec{a} + v\vec{b} \right)$

(ii) $\vec{r} : [0,1] \times [0, 2\pi] \longrightarrow \mathbb{R}^3$

$$(u,v) \longmapsto \vec{r}(u,v) = \begin{pmatrix} u \cos(v) \\ u \sin(v) \\ u \end{pmatrix}$$



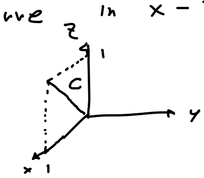
a cone!

Rem.: Usually we use r, φ as parameters: $\vec{r}(r, \varphi) = \begin{pmatrix} r \cos(\varphi) \\ r \sin(\varphi) \\ r \end{pmatrix}$

(iii) **Surfaces of revolution:**

Take a curve in $x - z$ - plane: $\vec{r} : [0,1] \longrightarrow \mathbb{R}^3$

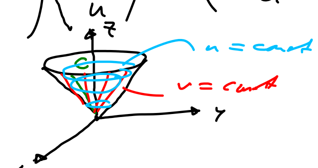
$$u \longmapsto \vec{r}(u) = \begin{pmatrix} u \\ 0 \\ u \end{pmatrix}$$



Then rotate around z - axis: $(v \in [0, 2\pi])$

$$\vec{r}(u,v) = \begin{pmatrix} \cos(v) & -\sin(v) & 0 \\ \sin(v) & \cos(v) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ 0 \\ u \end{pmatrix} = \begin{pmatrix} u \cos(v) \\ u \sin(v) \\ u \end{pmatrix}$$

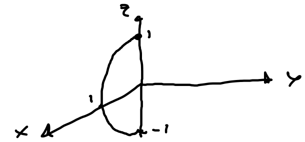
again our cone:



(iv) **Sphere:**

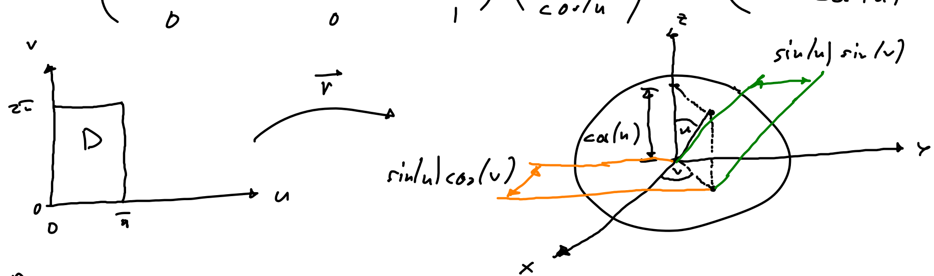
Consider the curve in $x - z$ - plane: $\vec{r} : [0, \pi] \longrightarrow \mathbb{R}^3$

$$u \longmapsto \vec{r}(u) = \begin{pmatrix} \sin(u) \\ 0 \\ \cos(u) \end{pmatrix}$$



rotation : $\left(v \in [0, 2\pi] \right)$

$$\vec{r}(u,v) = \begin{pmatrix} \cos(v) & -\sin(v) & 0 \\ \sin(v) & \cos(v) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sin(u) \\ 0 \\ \cos(u) \end{pmatrix} = \begin{pmatrix} \sin(u)\cos(v) \\ \sin(u)\sin(v) \\ \cos(u) \end{pmatrix}$$



Rem.:

$$\vec{r}(\theta, \varphi) = \begin{pmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{pmatrix}$$

θ : polar angle

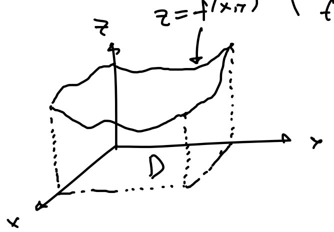
φ : azimuthal angle

(v) Graphs:

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x,y) \longmapsto z = f(x,y)$

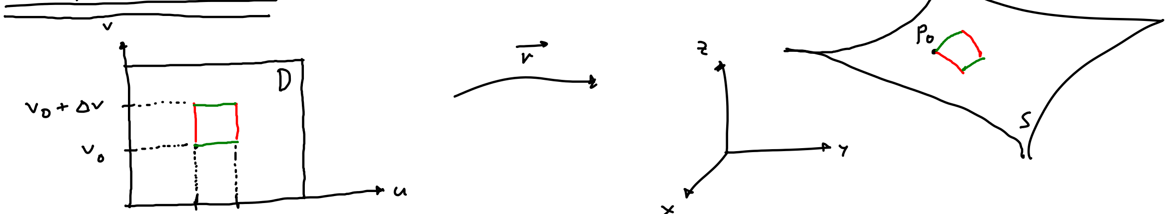
Then: $\vec{r}(x,y) = \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$ is a surface : the graph of f.



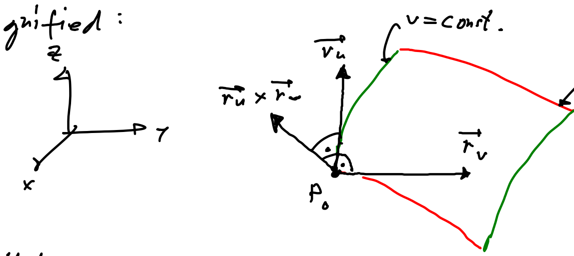
(everything in one picture!)

Surface Area

$P_0 = \vec{r}(u_0, v_0)$



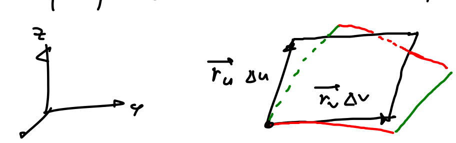
Patch magnified:



$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u}$$

$$\vec{r}_v = \frac{\partial \vec{r}}{\partial v}$$

Approx. of patch element by parallelogram:



$\longrightarrow$ Area of patch element:

$$\Delta A \approx \left| \vec{r}_u \Delta u \times \vec{r}_v \Delta v \right|$$

$$= \left| \vec{r}_u \times \vec{r}_v \right| \Delta u \, \Delta v$$

$\longrightarrow$ Total area $\approx \sum \left| \vec{r}_u \times \vec{r}_v \right| \Delta u \, \Delta v$

$$\boxed{\text{Area:} \quad A = \iint_D |\vec{r}_u \times \vec{r}_v| \, du\, dv \, .}$$

Ex: Surface of cone:

$$\vec{r}(u,v) = \begin{pmatrix} u\cos(v) \\ u\sin(v) \\ u \end{pmatrix} \longrightarrow \vec{r}_u = \frac{\partial \vec{r}}{\partial u} = \begin{pmatrix} \cos(v) \\ \sin(v) \\ 1 \end{pmatrix}$$

$$\vec{r}_v = \frac{\partial \vec{r}}{\partial v} = \begin{pmatrix} -u\sin(v) \\ u\cos(v) \\ 0 \end{pmatrix}$$

$$\longrightarrow |\vec{r}_u \times \vec{r}_v| = \left| \begin{pmatrix} \cos(v) \\ \sin(v) \\ 1 \end{pmatrix} \times \begin{pmatrix} -u\sin(v) \\ u\cos(v) \\ 0 \end{pmatrix} \right| = \left| \begin{pmatrix} -u\cos(v) \\ -u\sin(v) \\ u\cos^2(v) + u\sin^2(v) \end{pmatrix} \right|$$

$$= u\sqrt{2}$$

$$\longrightarrow A = \iint_D |\vec{r}_u \times \vec{r}_v| \, du\, dv = \int_{v=0}^{2\pi} \int_{u=0}^{1} \sqrt{2}\, u \, du\, dv = \dots = \sqrt{2}\,\pi$$