zu $\boxed{42}$ $\quad \dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$ (*) $\qquad$ Ges: $y(t,x)$: Auslenkung Seite

(a^2: Konst.)

An beiden Enden fixiert: $y(t,0)=0$, $y(t,L)=0$

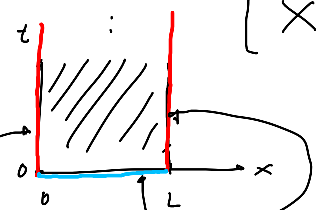
Zur Zeit $t=0$ ausgelenkt: $y(0,x) = f(x)$ gegeben.

Zur Zeit $t=0$ losgelassen: $\dfrac{\partial y}{\partial t}(0,x) = 0$.

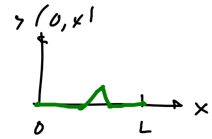
(i) Ansatz: $y(t,x) = T(t)\,X(x)$ in (*):

$$X T'' = a^2 X'' T$$

i.e. $\underbrace{\dfrac{T''}{a^2 T}}_{\text{hängt nur von } t \text{ ab}} = \underbrace{\dfrac{X''}{X}}_{\text{hängt nur von } x \text{ ab}} = \lambda$ (Konst.) $\longrightarrow \begin{cases} T'' = \lambda a^2 T \\ X'' = \lambda X \end{cases}$ 2 gewöhnliche DGL



(ii) $y(t,0) = y(t,L) = 0$ $\qquad y(0,x) = f(x)$

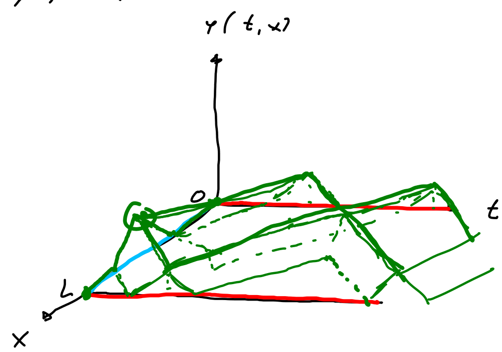


$y(t,0) \overset{!}{=} 0$

$\quad \|$

$T(t)X(0) = 0 \longrightarrow X(0) = 0$

analog: $\quad X(L) = 0$ $\qquad$ (**)



(iii) $X'' = \lambda X$:

$\lambda = 0 \longrightarrow X'' = 0 \longrightarrow X(x) = Ax + B$ $\quad$ ↯ inkomp. mit (**)

$\underset{\alpha^2}{\lambda} > 0 \longrightarrow X'' = \alpha^2 X \longrightarrow X(x) = A e^{\alpha x} + B e^{-\alpha x}$ $\quad$ ↯ inkomp. mit (**)

$\longrightarrow$ Nur $\underset{-\alpha^2}{\lambda} < 0$ möglich: $X'' = -\alpha^2 X$

$\longrightarrow X(x) = A\cos(\alpha x) + B\sin(\alpha x)$

$\longrightarrow X(0) = A \overset{!}{=} 0 \longrightarrow \underline{A = 0}$

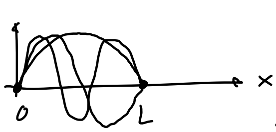
$X(L) = B\sin(\alpha L) \overset{!}{=} 0$

$\longrightarrow \sin(\alpha L) = 0$

$\longrightarrow \alpha L = n\pi \; ; \; n = \cancel{0}, \pm 1, \pm 2; \ldots$

$\longrightarrow \underline{\lambda} = -\alpha^2 = \underline{-\dfrac{n^2\pi^2}{L^2} \; ; \; n = 1, 2, 3, \ldots}$

[$X'' = \lambda X$



]

$$\longrightarrow \underline{X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right) \; ; \; n = 1, 2, 3, \ldots}$$

(iv) Anfangsbed: $\frac{\partial y}{\partial t}(0,x) = T'(0)X(x) \overset{!}{=} 0 \longrightarrow T'(0) = 0$

(v) $T'' = \lambda a^2 T \quad ; \quad \lambda = -\frac{n^2\pi^2}{L^2}$

$$\longrightarrow T'' = -\underbrace{\frac{n^2\pi^2 a^2}{L^2}}_{=\beta^2} T \qquad \text{I.e.} \quad T'' = -\beta^2 T$$

$\longrightarrow T(t) = A\cos(\beta t) + B\sin(\beta t)$

$\longrightarrow T'(t) = -A\beta\sin(\beta t) + B\beta\cos(\beta t)$

$\longrightarrow T'(0) = B\beta \overset{!}{=} 0 \longrightarrow \underline{B = 0}$

$\longrightarrow \underline{T_n(t) = A_n \cos\left(\frac{n\pi a}{L}t\right);\ n = 1,2,3,\ldots}$

(vi) $\underline{y_n(t,x)} = T_n(t)\,X_n(x)$

$$= \underline{\underbrace{A_n B_n}_{=C_n} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi a t}{L}\right)}$$

(vii) $y(t,x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi a t}{L}\right)$ ist auch Lösg., da $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$ Linear

$$\longrightarrow y(0,x) = \underline{\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)} \overset{!}{=} \underline{f(x)}$$

Anfangsbed.

$\longrightarrow C_n$ sind Koeff. der Fourier-Sinus-Reihe von $f(x)$:

$$\underline{C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

$$\longrightarrow \underline{y(t,x) = \sum_{n=1}^{\infty} \underbrace{\left(\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right)}_{C_n} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi a t}{L}\right)}$$