

# Lösung der Wärmeleitungsgleichung

$$\vec{q} = -k \nabla u \quad (\text{Fourier})$$

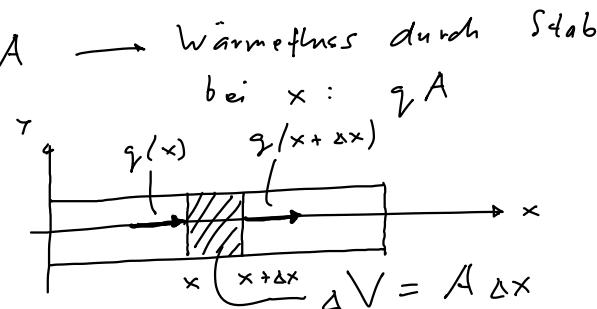
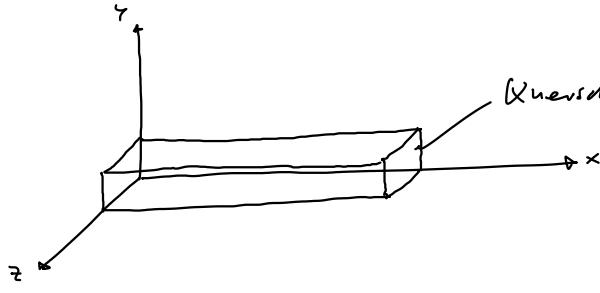
Temp.

Wärmefluss pro Fläche

Wärmeleitfähigkeit  $[k] = \frac{W}{mK}$

in 1d:  $q = -k \frac{\partial u}{\partial x}$

$$[q] = \frac{W}{mK} \frac{K}{m} = \frac{W}{m^2}$$



Innere (thermische) Energie in  $\Delta V$ :

$$Q = \rho \Delta V c u = \rho A \Delta x c u$$

$$\left. \begin{aligned} [\rho] &= \frac{kg}{m^3} \\ [c] &= \frac{J}{kg K} \end{aligned} \right\} \rightarrow [Q] = \frac{kg}{m^3} \frac{J}{kg K} m^3 K = J$$

Wärmefluss in  $\Delta V$ :  $-A(q(x+\Delta x) - q(x))$

Energieerhaltung:  $\frac{\partial Q}{\partial t} = -A(q(x+\Delta x) - q(x))$

$$\rightarrow \frac{\partial}{\partial t} (\rho A \Delta x c u) = -A(q(x+\Delta x) - q(x))$$

$$\begin{aligned} \rho c \frac{\partial u}{\partial t} &= - \frac{q(x+\Delta x) - q(x)}{\Delta x} \\ &= k \frac{\frac{\partial u}{\partial x}(x+\Delta x) - \frac{\partial u}{\partial x}(x)}{\Delta x} \end{aligned} \quad \left. \begin{aligned} q &= -k \frac{\partial u}{\partial x} \quad (\text{Fourier}) \\ \Delta x \rightarrow 0 &\rightarrow \frac{\partial^2 u}{\partial x^2} \end{aligned} \right\}$$

$$\rightarrow \rho c \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

mit:  $a^2 = \frac{k}{\rho c}$

$$\boxed{\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}}$$

Partielle Differenzialgl.:

Haben folgendes Problem:

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad \text{Längsstab} \quad (1) \\ u(t, 0) = u(t, L) = 0 : \text{Beide Enden des Stabes auf Nulltemp. Randbed.} \\ u(0, x) = f(x) : \text{Temp.-Verteilung zum Zeitpkt. } t = 0 \end{array} \right.$$

Ansatz: [Separation der Variablen]

$$\begin{aligned}
 u(t, x) &= T(t) X(x) \quad \text{in (1)} \\
 \rightarrow \frac{\partial}{\partial t} (T(t) X(x)) &= a^2 \frac{\partial^2}{\partial x^2} (T(t) X(x)) \\
 \rightarrow \frac{\partial T}{\partial t} (t) X(x) &= a^2 T(t) \frac{\partial^2 X}{\partial x^2} (x) \quad (*) \\
 \text{Mit: } \frac{\partial T}{\partial t} (t) &= \frac{d T}{d t} (t) = T'(t) \\
 \frac{\partial^2 X}{\partial x^2} (x) &= \frac{d^2 X}{d x^2} (x) = X''(x) \\
 \rightarrow (*) \text{ wird zu: } T' X &= a^2 T X'' \\
 \rightarrow \frac{T'}{a^2 T} &= \frac{X''}{X} \Rightarrow (\text{Konst.}) \\
 &\quad \begin{array}{l} \text{hängt von} \\ \text{nur von} \\ t \text{ ab} \end{array} \quad \begin{array}{l} \text{hängt von} \\ \text{nur von} \\ x \text{ ab} \end{array} \\
 \rightarrow \text{Haben 2 DGL:} & \boxed{X'' = \lambda X \quad (2) \\ T' = a^2 \lambda T \quad (3)}
 \end{aligned}$$

$$\begin{aligned}
 \text{Lösung: } X'' &= \lambda X \\
 \text{Mit } u(t, 0) &= u(t, L) = 0 \\
 &\quad \begin{array}{ll} \parallel & \parallel \\ T(t) X(0) & T(t) X(L) \end{array} \\
 & \quad \begin{array}{l} T(t) X(0) = 0 \\ T(t) X(L) = 0 \end{array} \quad \begin{array}{l} X(0) = 0 \\ X(L) = 0 \end{array} \\
 \text{I.e. haben Randwertproblem:} & \left\{ \begin{array}{l} X'' = \lambda X \\ X(0) = X(L) = 0 \end{array} \right.
 \end{aligned}$$

$\frac{3}{(i)} \underline{\lambda = 0} \rightarrow X'' = 0 \rightarrow X(x) = A x + B$   
 $X(0) = B \stackrel{!}{=} 0$ ,  
 $X(L) = A L \stackrel{!}{=} 0 \rightarrow A = 0$   
 $\rightarrow X(x) = 0 \rightarrow \text{irr inkompatibel}$   
 mit  $u(0, x) = f(x) \neq 0$

$$\begin{aligned}
 (ii) \underline{\lambda > 0} \quad \text{Sei } \alpha^2 &= \lambda \\
 X'' = \alpha^2 X &\rightarrow X(x) = A e^{\sqrt{\lambda} x} + B e^{-\sqrt{\lambda} x} \\
 &\rightarrow X(0) = A + B \stackrel{!}{=} 0 \rightarrow A = -B \\
 X(L) &= A e^{\sqrt{\lambda} L} + B e^{-\sqrt{\lambda} L} \\
 &= A (e^{\sqrt{\lambda} L} - e^{-\sqrt{\lambda} L}) \stackrel{!}{=} 0 \\
 &\rightarrow e^{\sqrt{\lambda} L} = e^{-\sqrt{\lambda} L} \\
 &\rightarrow 1 = e^{-2\sqrt{\lambda} L} \\
 &\rightarrow \sqrt{\lambda} L = 0 \rightarrow x = 0
 \end{aligned}$$

(iii)  $\lambda < 0$ : Sei  $\omega^2 = -\lambda$

$$X'' = -\omega^2 X \rightarrow X(x) = A \cos(\omega x) + B \sin(\omega x)$$

$$X(0) = A \stackrel{!}{=} 0 \rightarrow \underline{A = 0}$$

$$X(L) = B \sin(\omega L) \stackrel{!}{=} 0$$

$$\rightarrow \sin(\omega L) = 0$$

→ Bedingung an  $\omega$ :

$$\omega L = n\pi$$

$$\rightarrow \omega = \frac{n\pi}{L}; n=1, 2, 3, \dots$$

$$\rightarrow \lambda = -\omega^2 = -\frac{n^2 \pi^2}{L^2} ; n=1, 2, \dots$$

→ Lös. von  $X'' = \lambda X$  ist:

$$\boxed{X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right); n=1, 2, \dots}$$

I.e. haben  $\omega$ -vielen Lös. der DGL  $X'' = \lambda X$

Lös. von  $T' = \omega^2 \lambda T$ : →

$$\boxed{T_n(t) = \frac{D_n e^{\omega^2 \lambda t}}{e^{-\frac{n^2 \pi^2 \omega^2}{L^2} t}}} \\ \lambda = -\frac{n^2 \pi^2}{L^2}$$

$$\rightarrow u_n(t, x) = T_n(t) X_n(x)$$

$$= C_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2 \pi^2 \omega^2}{L^2} t}$$

$$\text{Basis } u_n(t, x) \text{ erfüllt } u(t, 0) = u(t, L) = 0 \\ (\text{Randbed.})$$

→ Auch Lin.-Kombi von  $u_n$  erfüllt Randbed.

Anfangsbed:  $u(0, x) = f(x)$  :

$$\exp(x) = e^x$$

$$\text{Versuchen: } u(t, x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2 \pi^2 \omega^2}{L^2} t\right)$$

$$\rightarrow u(0, x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{Fourier-Sinus-Reihe} \quad |||$$

$$\rightarrow \text{suchen } C_n, \text{ so dass: } u(0, x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

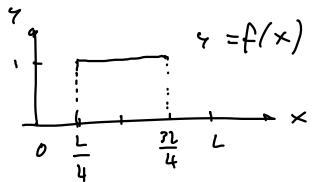
I.e. suchen Koeff. einer Fourier-Sinus-Reihe!

$$\rightarrow C_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

↓  
Anfangs-Temp.-Verteilung

$$\rightarrow u(t, x) = \sum_{n=1}^{\infty} \left( \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \right) \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2 a^2}{L^2} t\right)$$

Bsp:



$$f(x) = \begin{cases} 0 & : 0 \leq x < \frac{L}{4} \\ 1 & : \frac{L}{4} \leq x < \frac{3L}{4} \\ 0 & : \frac{3L}{4} \leq x < L \end{cases}$$

$$\rightarrow C_n = \frac{2}{L} \int_{\frac{L}{4}}^{\frac{3L}{4}} \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \left[ -\frac{\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}} \right]_{\frac{L}{4}}^{\frac{3L}{4}} = \frac{2}{n\pi} \left( \cos\left(\frac{n\pi}{4}\right) - \cos\left(\frac{3n\pi}{4}\right) \right)$$

$$\rightarrow u(t, x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \left( \cos\left(\frac{n\pi}{4}\right) - \cos\left(\frac{3n\pi}{4}\right) \right) \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2 a^2}{L^2} t\right)$$