<u>Wiederholung</u>

Sei $f(x)$ eine 2π-periodische Fkt.

<u>Fourierreihe:</u> $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

$$\left.\begin{matrix} a_0, a_1, a_2, \ldots \\ b_1, b_2, \ldots \end{matrix}\right\} \text{ <u>Koeff.</u>}$$

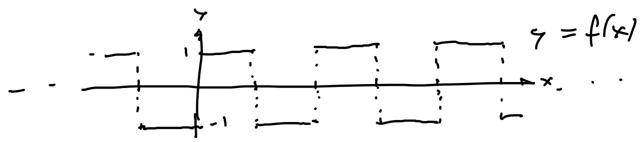
$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx)\,dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx)\,dx$$

<u>Bsp.:</u> (Rechteck)

$f(x)$ 2π-per. Fkt. mit $f(x) = \begin{cases} 1 & : 0 \leq x < \pi \\ -1 & \pi \leq x < 2\pi \end{cases}$



$$\boxed{a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx = 0}$$

(<u>Bem.:</u> $\frac{a_0}{2}$ = Mittelwert (DC-Anteil))

$$\boxed{a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx)\,dx}$$

$$= \frac{1}{\pi}\left(\int_0^{\pi} \cos(nx)\,dx - \int_{\pi}^{2\pi} \cos(nx)\,dx\right) = \ldots = \boxed{0}$$

$$\boxed{b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx)\,dx} = \frac{1}{\pi}\left(\int_0^{\pi} \sin(nx)\,dx - \int_{\pi}^{2\pi} \sin(nx)\,dx\right)$$

$$= \ldots = \boxed{\begin{cases} 0 & : n \text{ gerade} \\ \frac{4}{n\pi} & : n \text{ ung.} \end{cases}}$$

$$\longrightarrow f(x) = \overset{0}{\overset{\shortparallel}{\frac{a_0}{2}}} + \sum_{n=1}^{\infty} \left(\overset{0}{\overset{\shortparallel}{a_n}} \cos(nx) + b_n \sin(nx)\right)$$

$$= \frac{4}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \ldots\right)$$

<u>Bem.:</u> Können dies schreiben als:

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \underbrace{\frac{1}{2n-1}}_{} \sin((2n-1)x) = \frac{4}{\pi} \sum_{n \text{ ung.}} \frac{1}{n} \sin(nx).$$

↳ ergibt die ungeraden Zahlen.

<u>Aufgabe:</u> (Sägezahn)

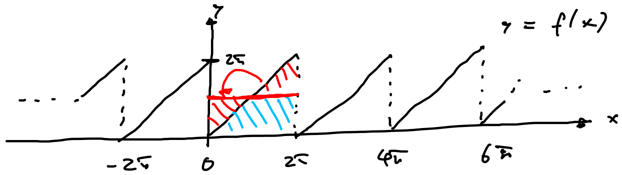
Sei $f(x)$ eine 2π-per. Fkt. mit $f(x) = x$ für $x \in [0, 2\pi)$

(i) Man skizziere $y = f(x)$

$$\left(f(x) = \begin{cases} x & : 0 \leq x < 2\pi \\ 2\pi\text{-per.} \end{cases} \right)$$

(ii) Man finde Fourierreihe von $f(x)$.

Lösg.: (i)



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

(ii) $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\cdot\frac{x^2}{2}\Big|_0^{2\pi} = 2\pi$$

$$\left[\ \bar{f} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx\ \right]$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left(\underbrace{x\frac{\sin(nx)}{n}\Big|_0^{2\pi}}_{=0} - \int_0^{2\pi}\frac{\sin(nx)}{n}\,dx\right)$$

$$= \frac{1}{\pi n^2}\cos(nx)\Big|_0^{2\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left(-\frac{x\cos(nx)}{n}\Big|_0^{2\pi} + \underbrace{\int_0^{2\pi}\frac{\cos(nx)}{n}\,dx}_{=0}\right)$$

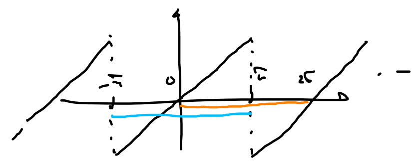
$$= \frac{1}{\pi}\left(-\frac{2\pi}{n}\right) = -\frac{2}{n}$$

$$\longrightarrow\ f(x) = \pi + \sum_{n=1}^{\infty}\left(-\frac{2}{n}\right)\sin(nx)$$

Bemerkung: Für $f(x)$ 2π-per. Fkt. gilt:

$$\int_0^{2\pi} f(x)\,dx = \int_I f(x)\,dx$$

I beliebiges Intervall der Länge 2π.



Bsp: $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

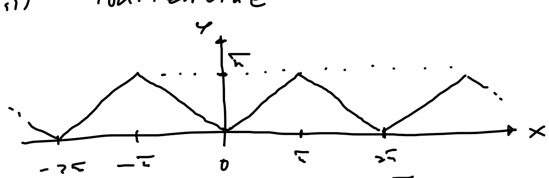
Aufgabe: Sei $f(x)$ 2π-per. Fkt. mit $f(x) = |x|$ für $x \in [-\pi, \pi)$.

(Dreieck) (i) Skizze von $y = f(x)$

(ii) Fourierreihe

$$\left[\begin{array}{ll} \text{Gerade:} & f(-x) = f(x) \\ \text{Ung.:} & f(-x) = -f(x) \end{array}\right]$$

Lösg.: (i)



$$|x| = \begin{cases} x : & x \geq 0 \\ -x : & x < 0 \end{cases}$$

(ii) $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0}(-x)\,dx + \int_0^{\pi} x\,dx\right)$$

$$= \frac{1}{\pi}\left(\left(-\frac{x^2}{2}\right)\Big|_{-\pi}^{0} + \frac{x^2}{2}\Big|_0^{\pi}\right)$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{2} + \frac{\pi^2}{2}\right) = \pi$$

$$\underline{a_n} = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0}(-x)\cos(nx)\,dx + \int_0^{\pi} x\cos(nx)\,dx\right)$$

$$= \frac{1}{\pi}\left(\underbrace{-\frac{x\sin(nx)}{n}\Big|_{-\pi}^{0}}_{=0} + \int_{-\pi}^{0}\frac{\sin(nx)}{n}\,dx + \underbrace{\frac{x\sin(nx)}{n}\Big|_0^{\pi}}_{=0} - \int_0^{\pi}\frac{\sin(nx)}{n}\,dx\right)$$

$$= \frac{1}{\pi}\left(-\frac{\cos(nx)}{n^2}\Big|_{-\pi}^{0} + \frac{\cos(nx)}{n^2}\Big|_0^{\pi}\right)$$

$$= \frac{1}{\pi n^2}\left(-1 + \cos(-n\pi) + \cos(n\pi) - 1\right)$$

$$= \frac{1}{\pi n^2}\left(-2 + 2\cos(n\pi)\right) = \frac{2}{\pi n^2}\left(-1 + \cos(n\pi)\right)$$

$$= \begin{cases} -\frac{4}{n^2\pi} & : n \text{ ung.} \\ 0 & : n \text{ ger.} \end{cases}$$

$$\underline{b_n} = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0}(-x)\sin(nx)\,dx + \int_0^{\pi} x\sin(nx)\,dx\right)$$

$$= \frac{1}{\pi}\left(\frac{x\cos(nx)}{n}\Big|_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\cos(nx)}{n}\,dx - \frac{x\cos(nx)}{n}\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos(nx)}{n}\,dx\right)$$

$$= \frac{1}{\pi}\left(\frac{\pi\cos(-n\pi)}{n} - \underbrace{\frac{\sin(nx)}{n^2}\Big|_{-\pi}^{0}}_{=0} - \frac{\pi\cos(n\pi)}{n} + \underbrace{\frac{\sin(nx)}{n^2}\Big|_0^{\pi}}_{=0}\right)$$

$$= \frac{1}{n}\left(\cos(-n\pi) - \cos(n\pi)\right) = \underline{0}$$

$$\longrightarrow \underline{f(x)} = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(x) + \frac{1}{9}\cos(3x) + \frac{1}{25}\cos(5x) + \dots\right)$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos((2n-1)x)$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ ung.}}\frac{1}{n^2}\cos(nx)$$