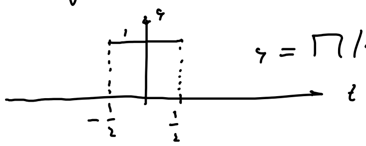
Wiederholung: Heuristischer Übergang: FR $\longrightarrow$ FT

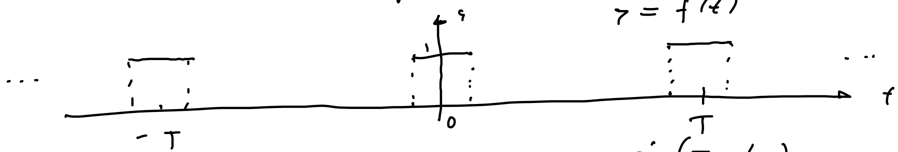
Bsp:

$y = \Pi(t)$



Period. Erweiterung, dann $T \longrightarrow \infty$

$y = f(t)$



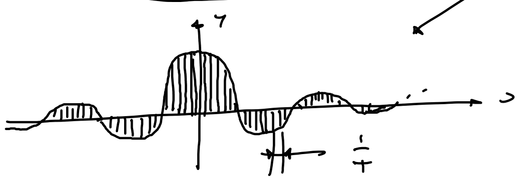
$\longrightarrow c_n = \frac{1}{\pi n} \sin\left(\frac{\pi n}{T}\right)$

Aber: $\lim\limits_{T \to \infty} c_n = 0$

$\longrightarrow$ betrachten: $T c_n = \frac{\sin(\pi n / T)}{\pi n / T}$

Mit $s = \frac{n}{T}$: $T c_n = \frac{\sin(\pi s)}{\pi s}$

diskret:



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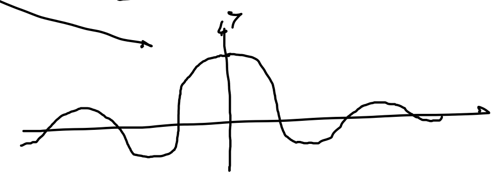


Mit Integral-Ausdruck für c_n:

$$c_n T = \int_{-T/2}^{T/2} f(t)\, e^{-j 2\pi n t / T}\, dt$$

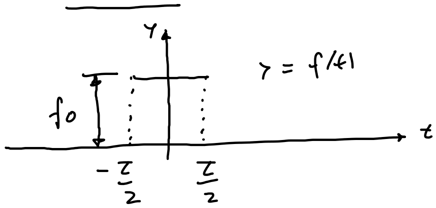
$T \longrightarrow \infty$, $s = \frac{n}{T}$:

$$\underline{\hat{\Pi}(s) = \int_{-\infty}^{\infty} \Pi(t)\, e^{-j 2\pi s t}\, dt}$$

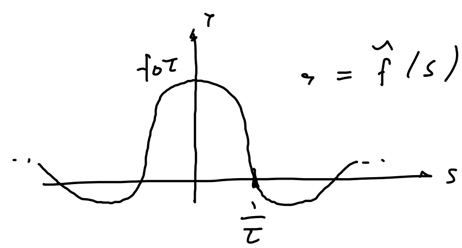
FT

Bsp: Knall:

$y = f(t)$



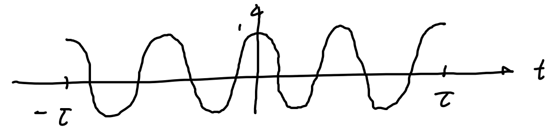
$y = \hat{f}(s) = f_0 \tau \operatorname{sinc}(s\tau)$



Breite Spektrum reziprok zu Knalldauer! |||

Bsp: Abgeschnittener Kosinus

$$f(t) = \begin{cases} \cos(\omega_0 t) & -\tau \le t \le t \\ 0 & \text{sonst.} \end{cases}$$



$$\hat{f}(s) = \int_{-\infty}^{\infty} f(t)\, e^{-j 2\pi s t}\, dt$$

$$= \int_{-\tau}^{\tau} \cos(\omega_0 t)\, e^{-j 2\pi s t}\, dt$$

$$= \int_{-\tau}^{\tau} \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\, e^{-j 2\pi s t}\, dt$$

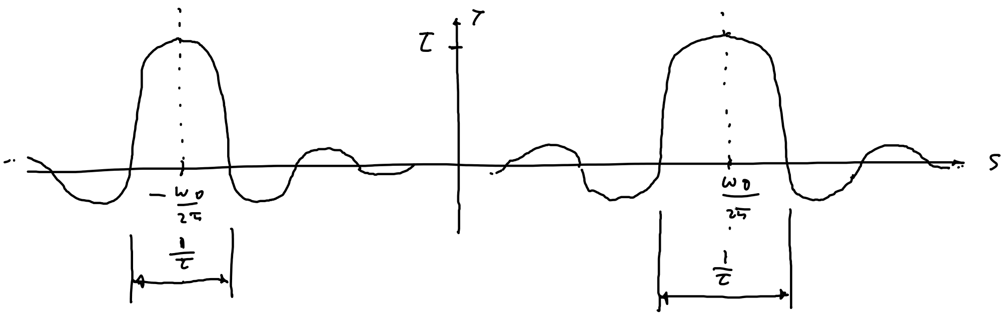
$$= \frac{1}{2}\left(\int_{-\tau}^{\tau} e^{j(\omega_0-2\pi s)t}\,dt + \int_{-\tau}^{\tau} e^{-j(\omega_0+2\pi s)t}\,dt\right)$$

$$= \frac{1}{2}\left(\frac{e^{j(\omega_0-2\pi s)t}}{j(\omega_0-2\pi s)}\Bigg|_{-\tau}^{\tau} + \frac{e^{-j(\omega_0+2\pi s)t}}{-j(\omega_0+2\pi s)}\Bigg|_{-\tau}^{\tau}\right)$$

$$= \frac{e^{j(\omega_0-2\pi s)\tau}-e^{-j(\omega_0-2\pi s)\tau}}{2j(\omega_0-2\pi s)} + \frac{e^{-j(\omega_0+2\pi s)\tau}-e^{j(\omega_0+2\pi s)\tau}}{-2j(\omega_0+2\pi s)}$$

$$= \frac{\sin((\omega_0-2\pi s)\tau)}{\omega_0-2\pi s} + \frac{\sin((\omega_0+2\pi s)\tau)}{\omega_0+2\pi s}$$

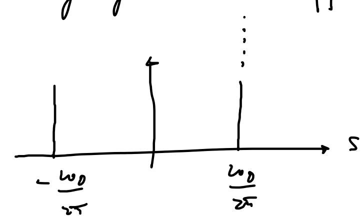
$$= \underline{\tau\,\mathrm{sinc}\left(\left(\tfrac{\omega_0}{\pi}-2s\right)\tau\right) + \tau\,\mathrm{sinc}\left(\left(\tfrac{\omega_0}{\pi}+2s\right)\tau\right)}$$



Breite Spektrum: $\frac{1}{\tau}$

$\longrightarrow$ Wieder reziprok zur Dauer des Vorgangs.

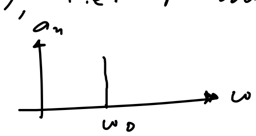
Für $\tau \longrightarrow \infty$: Scharfe Spektrallinien bei $s = \pm\frac{\omega_0}{2\pi}$



Macht Sinn!

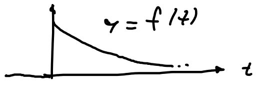
$\tau \to \infty$ bedeutet: $f(t) = \cos(\omega_0 t)$, i.e. periodisch!

$\longrightarrow$ Spektrum:



(Wir bekommen diskretes Spektrum, bis auf Vorfaktor τ).

Aufgabe: Man finde $\hat{f}(s)$ für $f(t) = \begin{cases} 0 & t < 0 \\ e^{-at} & t > 0 \end{cases}$ $(a>0)$

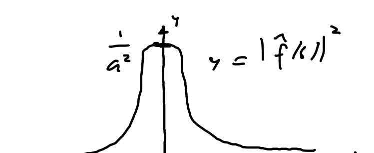


und skizziere $|\hat{f}(s)|^2$.

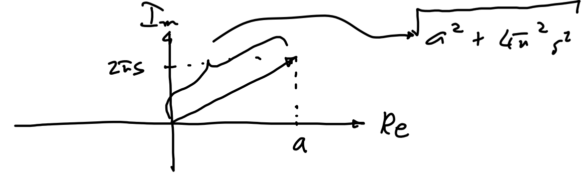
Lösg.: $\hat{f}(s) = \int_0^\infty e^{-at} e^{-j2\pi st}\,dt = \int_0^\infty e^{-(a+j2\pi s)t}\,dt$

$$= \frac{e^{-(a+j2\pi s)t}}{-(a+j2\pi s)}\Bigg|_0^\infty = \frac{1}{a+j2\pi s}$$

$$\longrightarrow |\hat{f}(s)|^2 = \underline{\underline{\frac{1}{|a+j2\pi s|^2}}} = \frac{1}{a^2+4\pi^2 s^2}$$

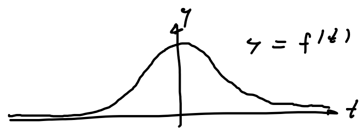


Leistungsspektrum



FT der Gausskurve

Sei $f(t) = e^{-\pi t^2}$



$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-j 2\pi s t} \, dt$$

$$\underline{\frac{d\hat{f}}{ds}}(s) = \frac{d}{ds} \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-j2\pi s t} dt = \int_{-\infty}^{\infty} e^{-\pi t^2} \frac{d}{ds}\left(e^{-j 2\pi s t}\right) dt$$

$$= \int_{-\infty}^{\infty} e^{-\pi t^2}(-j2\pi t)\, e^{-j2\pi s t} \, dt$$

$$= j \int_{-\infty}^{\infty} \underbrace{\overbrace{e^{-\pi t^2}}^{=f(t)}(-2\pi t)}_{=\frac{df}{dt}(t)} \underbrace{e^{-j2\pi s t}}_{=g(t)} dt$$

$$= j\left(\underbrace{e^{-\pi t^2} e^{-j2\pi s t}\Big|_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} e^{-\pi t^2}(-j 2\pi s) e^{-j2\pi s t} dt\right)$$

$$= -2\pi s \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-j2\pi s t} dt$$

$$= \underline{-2\pi s \, \hat{f}(s)}$$

I.e. $\underline{\frac{d\hat{f}}{ds}(s) = -2\pi s \, \hat{f}(s)}$

Separation: $\int \frac{d\hat{f}}{\hat{f}} = \int -2\pi s \, ds \longrightarrow \log(\hat{f}) = -\pi s^2 + C$

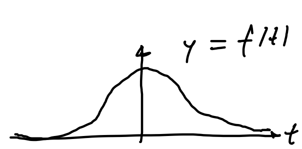
$\longrightarrow \hat{f}(s) = C e^{-\pi s^2}$

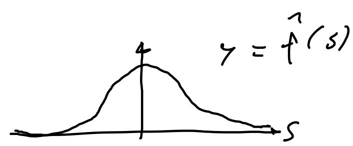
mit: $C = \hat{f}(0)$

$\longrightarrow \underline{\hat{f}(s) = \hat{f}(0)\, e^{-\pi s^2}}$

Was ist $\hat{f}(0)$?

Es gilt: $\hat{f}(0) = \int_{-\infty}^{\infty} e^{-\pi t^2} dt = 1 \longrightarrow \underline{\hat{f}(s) = e^{-\pi s^2}}$





I.e. <u>Gausskurve ist seine eigene FT!</u>

<u>Beweis:</u>

$$\int_{-\infty}^{\infty} e^{-\pi t^2}\,dt \underset{u=\sqrt{\pi}\,t}{=} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2}\,du = \frac{1}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-t^2}\,dt}_{I}$$

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-t^2}\,dt \right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx \right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy \right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} \cdot e^{-y^2}\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,\underline{dx\,dy}$$

$$= \int_{\varphi=0}^{2\pi}\int_{r=0}^{\infty} e^{-r^2}\,\underline{r\,dr\,d\varphi}$$

$$= \int_{\varphi=0}^{2\pi} -\frac{1}{2} e^{-r^2}\Big|_0^{\infty}\,d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \frac{1}{2}\,d\varphi = \underline{\pi}$$

$$\longrightarrow I^2 = \pi \longrightarrow I = \sqrt{\pi} \longrightarrow \underline{\int_{-\infty}^{\infty} e^{-\pi t^2}\,dt = 1} \qquad \square$$