Faltung

Problem: $F(s) = F_1(s)\, F_2(s)$

Gesucht: Rücktrafo: $f(t) = \mathcal{L}^{-1}(F(s))$

Bekannt sei zudem: $f_1(t) = \mathcal{L}^{-1}(F_1(s))$

$f_2(t) = \mathcal{L}^{-1}(F_2(s))$

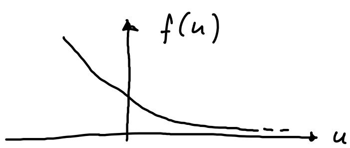
⚠ $f(t) \neq f_1(t)\, f_2(t)$

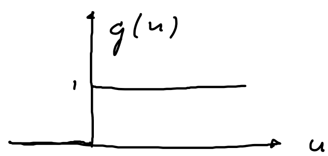
Jedoch gilt: $f(t) = (f_1 * f_2)(t)$

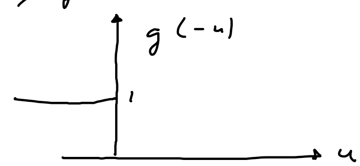
Def: Faltungsprodukt:

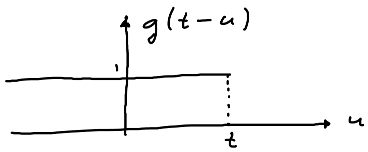
$$\boxed{(f_1 * f_2)(t) = \int_0^t f_1(u)\, f_2(t-u)\, du \qquad t \geq 0}$$

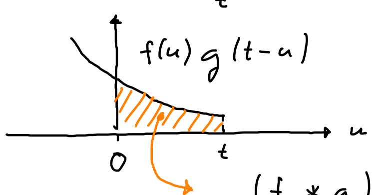
Grafische Interpretation: $f(u) = e^{-u}$; $g(u) = H(u)$ $(f * g)$







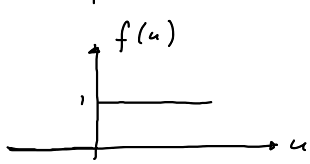


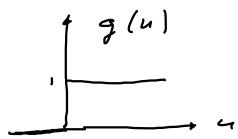


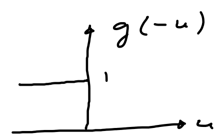
$$(f * g)(t) = \int_0^t f(u)\, g(t-u)\, du$$

Aufgabe: Grafische Interpretation von $(f * g)(t)$ mit:
$f(u) = H(u)$; $g(u) = H(u)$

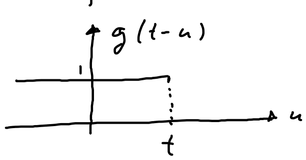
Lösg.:

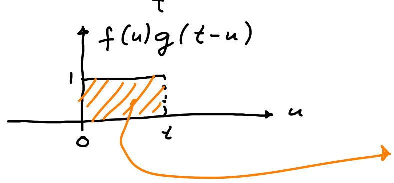




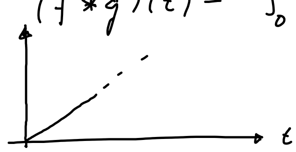


(Convolution)





$$(f * g)(t) = \int_0^t f(u)\, g(t-u)\, du$$



Eigenschaften:

(i) Kommutativ: $f_1 * f_2 = f_2 * f_1$

(ii) Assoziativ: $(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)$

(iii) Distributiv: $f_1 * (f_2 + f_3) = f_1 * f_2 + f_1 * f_3$

Bsp:

$$e^{at} * e^{bt} = \int_0^t e^{au} e^{b(t-u)} \, du$$

$$= \int_0^t e^{(a-b)u} e^{bt} \, du = e^{bt} \int_0^t e^{(a-b)u} \, du$$

$$= e^{bt} \frac{1}{a-b} e^{(a-b)u} \Big|_0^t = e^{bt} \frac{1}{a-b} \left(e^{(a-b)t} - 1 \right)$$

$$= \frac{e^{at} - e^{bt}}{a-b}$$

Aufgabe: Sei $f(t) = e^{at}$

Man bestimme $(f * f)(t)$

Lösg.: $\underline{(f*f)(t)} = \int_0^t e^{au} e^{a(t-u)} \, du = e^{at} \int_0^t \underbrace{e^{au-au}}_{=1} \, du = \underline{t e^{at}}$

Faltungssatz:

$$\boxed{\mathcal{L}(f_1 * f_2) = \mathcal{L}(f_1)\,\mathcal{L}(f_2)}$$

Als Diagramm:

$$\boxed{\begin{tikzcd}
f_1(t) \arrow[r, "\mathcal{L}"] \arrow[d] & F_1(s) \arrow[d] \\
(f_1 * f_2)(t) \arrow[r, "\mathcal{L}"] & F_1(s) F_2(s) \\
f_2(t) \arrow[r, "\mathcal{L}"] \arrow[u] & F_2(s) \arrow[u]
\end{tikzcd}}$$

Bsp: Sei $F(s) = \frac{1}{(s^2+1)s}$

Ges: $f(t) = \mathcal{L}^{-1}(F(s))$ mit Faltungssatz

$$F(s) = \underbrace{\frac{1}{s^2+1}}_{=F_1(s)} \cdot \underbrace{\frac{1}{s}}_{=F_2(s)}$$

$$\begin{tikzcd}
\sin(t) \arrow[r, "\mathcal{L}"] \arrow[d] & \frac{1}{s^2+1} \arrow[d] \\
\sin(t) * 1 \arrow[r, "\mathcal{L}"] & \frac{1}{(s^2+1)s} \\
1 \arrow[r, "\mathcal{L}"] \arrow[u] & \frac{1}{s} \arrow[u]
\end{tikzcd}$$

Haben: $f_1(t) = \mathcal{L}^{-1}(F_1(s)) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin(t)$

$f_2(t) = \mathcal{L}^{-1}(F_2(s)) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$

$\longrightarrow$ $\underline{f(t)} = \mathcal{L}^{-1}(F_1(s) F_2(s)) = (f_1 * f_2)(t)$

$$= \int_0^t \underbrace{f_1(u)}_{=\sin(u)} \underbrace{f_2(t-u)}_{=1} du = \int_0^t \sin(u) \, du = -\cos(u) \Big|_0^t$$

$$= -\cos(t) - (-\cos(0))$$

$$= \underline{-\cos(t) + 1}$$

I.e. $\underline{\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)s}\right) = -\cos(t) + 1}$

Aufgabe: Man finde $\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)s^2}\right)$ mit Faltungssatz.

Lösg.: $F(s) = \underbrace{\frac{1}{s^2+1}}_{=F_1(s)} \cdot \underbrace{\frac{1}{s^2}}_{=F_2(s)}$ $\longrightarrow$ $f_1(t) = \mathcal{L}^{-1}(F_1(s)) = \sin(t)$

$f_2(t) = \mathcal{L}^{-1}(F_2(s)) = t$

$\longrightarrow$ $\underline{f(t)} = (f_1 * f_2)(t) = \int_0^t f_1(u) f_2(t-u) \, du$

$$
\begin{aligned}
&= \int_0^t \sin(u)\,(t-u)\,du \\
&= t\int_0^t \sin(u)\,du - \int_0^t u\sin(u)\,du \\
&= t\left(-\cos(t)+1\right) - \left(-u\cos(u)\Big|_0^t + \int_0^t \cos(u)\,du\right) \\
&= t\left(-\cos(t)+1\right) - \left(-t\cos(t) + \sin(t)\right) \\
&= \underline{t - \sin(t)}
\end{aligned}
$$

<u>Bsp:</u> Anfangswertproblem mit beliebiger Störfkt.:

$$
\left.\begin{aligned}
4y'' + y &= g(t) \\
y(0) &= 3 \\
y'(0) &= -7
\end{aligned}\right\}
\qquad
\begin{aligned}
\mathcal{L}(y) &= Y \\
\mathcal{L}(y') &= sY - y(0) = sY - 3 \\
\mathcal{L}(y'') &= s(sY-3) - y'(0) \\
&= s^2Y - 3s + 7
\end{aligned}
$$

$\longrightarrow$ DGL wird zu: $4(s^2Y - 3s + 7) + Y = \underbrace{G(s)}_{=\mathcal{L}(g(t))}$

$$Y(4s^2+1) = 12s - 28 + G$$

$$\longrightarrow Y = \frac{12s-28}{4s^2+1} + \frac{G}{4s^2+1}$$

<u>$\mathcal{L}^{-1}\left(\frac{12s-28}{4s^2+1}\right)$:</u>

$$
\begin{tikzcd}
\frac{1}{4}\cos(t) \arrow[r, "\mathcal{L}"] \arrow[d] & \frac{1}{4}\cdot\frac{s}{s^2+1} \arrow[d] \\
\frac{1}{4}\cos\left(\frac{t}{2}\right) \arrow[r, "\mathcal{L}"] & \frac{s}{4s^2+1} = \frac{s}{(2s)^2+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\frac{1}{2}\sin(t) \arrow[r, "\mathcal{L}"] \arrow[d] & \frac{1}{2}\cdot\frac{1}{s^2+1} \arrow[d] \\
\frac{1}{2}\sin\left(\frac{t}{2}\right) \arrow[r, "\mathcal{L}"] & \frac{1}{4s^2+1} = \frac{1}{(2s)^2+1}
\end{tikzcd}
$$

$$\longrightarrow \mathcal{L}^{-1}\left(\frac{12s-28}{4s^2+1}\right) = 3\cos\left(\frac{t}{2}\right) - 14\sin\left(\frac{t}{2}\right)$$

<u>$\mathcal{L}^{-1}\left(\frac{G(s)}{4s^2+1}\right)$:</u> mit <u>Faltungssatz:</u>

$$
\begin{tikzcd}
g(t) \arrow[r, "\mathcal{L}"] \arrow[d] & G(s) \arrow[d] \\
\frac{1}{2}\sin\left(\frac{t}{2}\right) * g(t) \arrow[r] & \frac{G(s)}{4s^2+1} \\
\frac{1}{2}\sin\left(\frac{t}{2}\right) \arrow[r, "\mathcal{L}"] \arrow[u] & \frac{1}{4s^2+1} \arrow[u]
\end{tikzcd}
$$

I.e. $\mathcal{L}^{-1}\left(\frac{G(s)}{4s^2+1}\right) = \int_0^t \frac{1}{2}\sin\left(\frac{u}{2}\right) g(t-u)\,du$

$$\longrightarrow \underline{y(t) = 3\cos\left(\frac{t}{2}\right) - 14\sin\left(\frac{t}{2}\right) + \int_0^t \frac{1}{2}\sin\left(\frac{u}{2}\right) g(t-u)\,du}$$

Faltung mit δ- Fkt.: $\quad (\delta * f)(t) = \int_0^t \delta(u)\, f(t-u)\, du$

$$= f(t)$$

$$\longrightarrow \quad \underline{\delta * f = f * \delta = f}$$

I.e. δ ist Identität des Faltungsprodukts.

Aufgabe: Ges.: $\mathcal{L}^{-1}\left(\frac{1}{s^2-4}\right)$ mit Faltungssatz

$$\left(\text{Idee: } \frac{1}{s^2-4} = \underbrace{\frac{1}{s+2}}_{F_1} \cdot \underbrace{\frac{1}{s-2}}_{F_2} \right)$$

Aufgabe: (zur δ-Fkt.)

$$\left.\begin{aligned} y'' + 3y' + 2y &= 1 + \delta(t-4) \\ y(0) = y'(0) &= 0 \end{aligned}\right\}$$