Aufgabe: (i) $\left.\begin{aligned} y'' + 2y' + y &= 0 \\ y(0) &= 0 \\ y'(0) &= 1 \end{aligned}\right\}$

$$\mathcal{L}(y) = Y$$
$$\mathcal{L}(y') = sY - y(0) = sY$$
$$\mathcal{L}(y'') = s^2 Y - y'(0) = s^2 Y - 1$$

$\longrightarrow$ DGL wird zu: $s^2 Y - 1 + 2sY + Y = 0$

I.e. $Y(s^2 + 2s + 1) = 1 \longrightarrow Y(s) = \frac{1}{s^2+2s+1} = \frac{1}{(s+1)^2}$

$\longrightarrow \underline{y(t)} = \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) = \underline{te^{-t}}$

$$\begin{tikzcd} t \arrow[r, "\mathcal{L}"] \arrow[d, "e^{-t}"'] & \frac{1}{s^2} \arrow[d, "s \mapsto s+1"] \\ te^{-t} \arrow[r, "\mathcal{L}"] & \frac{1}{(s+1)^2} \end{tikzcd}$$

(ii) $\left.\begin{aligned} y' + 2y &= 4t \\ y(0) &= 1 \end{aligned}\right\}$

$$\mathcal{L}(y) = Y$$
$$\mathcal{L}(y') = sY - y(0) = sY - 1$$
$$\mathcal{L}(4t) = \frac{4}{s^2}$$

$\longrightarrow$ DGL wird zu: $sY - 1 + 2Y = \frac{4}{s^2}$

I.e. $Y(s+2) = \frac{4}{s^2} + 1 \longrightarrow Y(s) = \frac{4}{s^2(s+2)} + \frac{1}{s+2}$

PBZ: $\frac{4}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} \longrightarrow \dots \longrightarrow \begin{aligned} A &= -1 \\ B &= 2 \\ C &= 1 \end{aligned}$

$$\longrightarrow Y(s) = -\frac{1}{s} + \frac{2}{s^2} + \frac{1}{s+2} + \frac{1}{s+2}$$
$$= -\frac{1}{s} + \frac{2}{s^2} + \frac{2}{s+2}$$

$$\longrightarrow \underline{y(t)} = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(-\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{2}{s^2}\right) + \mathcal{L}^{-1}\left(\frac{2}{s+2}\right)$$
$$= \underline{-1 + 2t + 2e^{-2t}}$$

Alternative zu (ii):

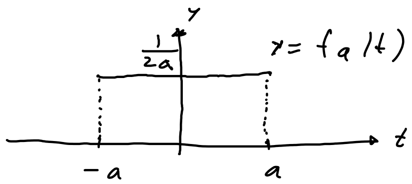
$$\begin{tikzcd} e^{-2t} \arrow[r, "\mathcal{L}"] \arrow[d, "\int_0^t \cdots"'] & \frac{1}{s+2} \arrow[d, "\cdot \frac{1}{s}"] \\ -\frac{1}{2}e^{-2t} + \frac{1}{2} \arrow[r, "\mathcal{L}"] \arrow[d, "\int_0^t \cdots"'] & \frac{1}{s(s+2)} \arrow[d, "\cdot \frac{1}{s}"] \\ \underline{\frac{1}{4}e^{-2t} - \frac{1}{4} + \frac{1}{2}t} \arrow[r, "\mathcal{L}"] & \frac{1}{s^2(s+2)} \end{tikzcd}$$

$$\longrightarrow \mathcal{L}^{-1}\left(\frac{4}{s^2(s+2)}\right) = e^{-2t} - 1 + 2t$$

mit $\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = e^{-2t} \longrightarrow \underline{y(t) = 2e^{-2t} + 2t - 1}$

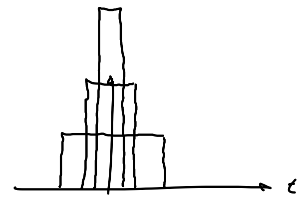
Dirac'sche Verteilung

Betrachten: $f_a(t) = \frac{1}{2a}\left(H(t+a) - H(t-a)\right)$ mit $a > 0$.



Unabhängig von a gilt:

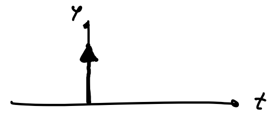
$$\int_{-\infty}^{\infty} f_a(t)\,dt = 1$$



Formale Def. der Dirac'schen δ - Fkt.:

$$\delta(t) = \lim_{a \to 0} f_a(t)$$

Notation:

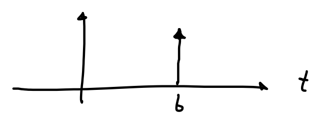


Es gilt: $\delta(t) = 0$ für $t \neq 0$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$\delta(t)$ ist keine Fkt. im herkömmlichen Sinn.

Verschiebung: $\delta(t-b)$:



$\delta(t-b) = 0$ für $t \neq b$.

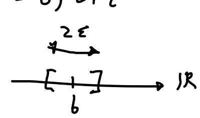
Fundamentale Eigenschaft:

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-b)\,dt = f(b) \quad (\text{für } f \text{ stetig})$$

Erklärung: (heuristisch)

$f(t)\,\delta(t-b) = 0$ für $t \neq b$

$$\longrightarrow \int_{-\infty}^{\infty} f(t)\,\delta(t-b)\,dt = \int_{b-\varepsilon}^{b+\varepsilon} f(t)\,\delta(t-b)\,dt \quad \text{für } \varepsilon \text{ beliebig klein.}$$



$f(t)$ stetig $\longrightarrow$ $f(t)$ im Intervall $[b-\varepsilon, b+\varepsilon]$ für ε sehr klein ist in etwa $f(b)$

$$\longrightarrow \int_{b-\varepsilon}^{b+\varepsilon} f(t)\,\delta(t-b)\,dt = f(b) \underbrace{\int_{b-\varepsilon}^{b+\varepsilon} \delta(t-b)\,dt}_{=1} = f(b).$$

Laplace-Trafo

$$\underline{\mathcal{L}(\delta(t))} = \int_0^\infty \delta(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} H(t)\, \delta(t)\, e^{-st}\, dt$$

$$= H(0) = \underline{1}$$

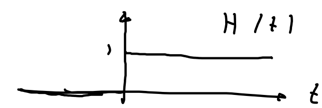
Für $b \geq 0$:

$$\underline{\mathcal{L}(\delta(t-b))} = \int_{-\infty}^{\infty} H(t)\, \delta(t-b)\, e^{-st}\, dt = \underline{e^{-sb}}$$

Wir wissen somit:

$$H(t) \xrightarrow{\mathcal{L}} \frac{1}{s}$$

$$\delta(t) \xrightarrow{\mathcal{L}} 1$$



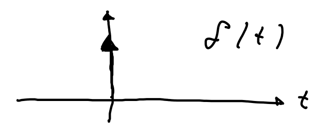


Diagramm: Integration Zeitbereich:

$$\begin{tikzcd}
\delta(t) \arrow[r, "\mathcal{L}"] \arrow[d, "\int_0^t \cdots"'] & 1 \arrow[d, "\cdot \frac{1}{s}"] \\
H(t) \arrow[r, "\mathcal{L}"] & \frac{1}{s}
\end{tikzcd}$$

"In diesem Sinn ist die δ-Fkt. die Ableitung der Heaviside-Fkt."

Bsp: $F(s) = \dfrac{s^2}{s^2+4} = 1 - \dfrac{4}{s^2+4}$

$$\longrightarrow \mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}(1) - \mathcal{L}^{-1}\left(\frac{4}{s^2+4}\right)$$

$$= \delta(t) - 2\sin(2t)$$

$$\left(\begin{tikzcd}
2\sin(t) \arrow[r, "\mathcal{L}"] \arrow[d] & \frac{2}{s^2+1} \arrow[d] \\
2\sin(2t) \arrow[r, "\mathcal{L}"] & \frac{4}{s^2+4} = \frac{1}{\left(\frac{s}{2}\right)^2+1}
\end{tikzcd}\right)$$

Wichtige Bemerkung:

Betrachte zwei Anfangswertprobleme:

(i) $\left.\begin{aligned} y'' + y &= 0 \\ y(0) &= 0 \\ y'(0) &= v_0 \end{aligned}\right\}$ z.B. Oszillator mit Anfangsgeschw.

(ii) $\left.\begin{aligned} y'' + y &= v_0\, \delta(t) \\ y(0) &= 0 \\ y'(0) &= 0 \end{aligned}\right\}$ z.B. Oszillator ohne Anfangsgeschw. auf welchen Kraft $v_0\, \delta(t)$ wirkt.

Laplace Trafo:

(i) $\mathcal{L}(y) = Y$

$\mathcal{L}(y') = sY - \overset{\overset{0}{\shortparallel}}{y(0)}$

$$\mathcal{L}(y'') = s^2 Y - \underset{\overset{\shortparallel}{v_0}}{y'(0)} = s^2 Y - v_0$$

$\longrightarrow$ DGL wird zu: $\quad s^2 Y - v_0 + Y = 0 \longrightarrow \underline{Y = \dfrac{v_0}{s^2+1}}$

(ii) $\mathcal{L}(y) = Y$

$\mathcal{L}(y') = sY$

$\mathcal{L}(y'') = s^2 Y - \underset{\overset{\shortparallel}{0}}{y'(0)}$

$\mathcal{L}(v_0 \delta(t)) = v_0$

$\longrightarrow$ DGL wird zu: $s^2 Y + Y = v_0$

$\longrightarrow \underline{Y = \dfrac{v_0}{s^2+1}}$

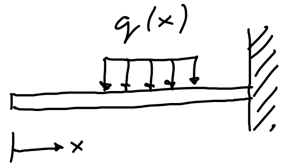
$\longrightarrow$ Laplace-Trafos stimmen überein!

$\longrightarrow$ Systeme verhalten sich gleich!

$\longrightarrow$ Physikalisch entspricht $v_0 \delta(t)$ einem Impuls zur Zeit $t = 0$!

Andere Anwendung von $\delta(x)$, $H(x)$:

Biegung:



Streckenlast [in $\frac{N}{m}$]

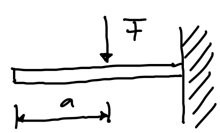
$\longrightarrow$ Scherkraft: $\quad -V(x) = \displaystyle\int_0^x q(u)\,du + C_1 \quad$ [in N]

$\longrightarrow$ Biegemoment: $\quad M(x) = \displaystyle\int_0^x -V(u)\,du \quad$ [in Nm]

Gl. der Biegelinie: $\boxed{EI\, y''(x) = M(x)}$

y: Auslenkung

Streckenlast für Einzelkraft bei $x = a$:



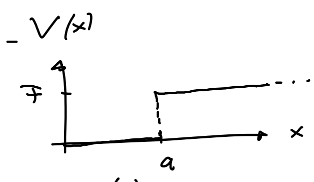
$$q(x) = F\,\delta(x-a)$$

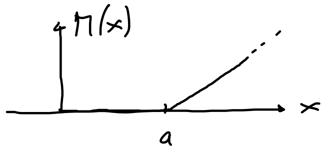
$$\longrightarrow -V(x) = \int_0^x F\,\delta(u-a)\,du$$

$$= F \int_0^x \delta(u-a)\,du$$

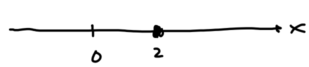
$$= \begin{cases} 0 & x < a \\ F & x \geq a \end{cases}$$

$$= F\, H(x-a)$$





Ladungsdichte Elektron: (in 1D)



$\rho(x)$: Ladungsdichte

$$\longrightarrow Q = \int_{-\infty}^{\infty} \rho(x)\,dx$$

$$\rho(x) = q\,\delta(x-2)$$

$$\rightarrow \underline{Q} = \int_{-\infty}^{\infty} q\, \delta(x-z)\, dx = \underline{q}$$