Wiederholung

Bsp: (i)

$$\mathcal{L}^{-1}\left(\frac{s}{s^2-1}\right) \overset{\text{PBZ}}{=} \mathcal{L}^{-1}\left(\frac{1}{2}\cdot\frac{1}{s-1}+\frac{1}{2}\cdot\frac{1}{s+1}\right)$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)+\frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$= \underline{\frac{1}{2}e^{t}+\frac{1}{2}e^{-t}}$$

(ii)

$$\mathcal{L}^{-1}\left(\frac{5s^2+2s+1}{s^3+s}\right) \underset{\text{PBZ}}{=} \mathcal{L}^{-1}\left(\frac{1}{s}\right)+\mathcal{L}^{-1}\left(\frac{2+4s}{s^2+1}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s}\right)+2\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)+4\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)$$

$$= \underline{1+2\sin(t)+4\cos(t)}$$

(iii)

$$\mathcal{L}^{-1}\left(\frac{2}{s^2+6s+13}\right) \underset{\text{quadratisch ergänzen}}{=} \mathcal{L}^{-1}\left(\frac{2}{(s+3)^2+4}\right) = e^{-3t}\sin(2t)$$

$$\begin{tikzcd}
\sin(t) \arrow[r, "\mathcal{L}"] \arrow[d, "t \mapsto 2t"'] & \frac{1}{s^2+1} \arrow[d] \\
\sin(2t) \arrow[r, "\mathcal{L}"] \arrow[d, "\cdot e^{-3t}"'] & \frac{2}{s^2+4} = \frac{\frac{1}{2}}{\left(\frac{s}{2}\right)^2+1} \arrow[d] \\
e^{-3t}\sin(2t) \arrow[r, "\mathcal{L}"] & \frac{2}{(s+3)^2+4}
\end{tikzcd}$$

Ähnlichkeit

Dämpfung

Bsp:

$$F(s) = e^{-2s}\,\frac{2}{s^2+9}$$

$$\begin{tikzcd}
\frac{2}{3}\sin(t) \arrow[r, "\mathcal{L}"] \arrow[d, "t \mapsto 3t"'] & \frac{\frac{2}{3}}{s^2+1} \arrow[d] \\
H(t)\frac{2}{3}\sin(3t) \arrow[r, "\mathcal{L}"] \arrow[d] & \frac{2}{s^2+9} = \frac{\frac{2}{9}}{\left(\frac{s}{3}\right)^2+1} \arrow[d, "\cdot e^{-2s}"] \\
H(t-2)\frac{2}{3}\sin(3(t-2)) = H(t-2)\frac{2}{3}\sin(3t-6) \arrow[r, "\mathcal{L}"] & e^{-2s}\frac{2}{s^2+9}
\end{tikzcd}$$

Ähnlichkeit

Verschiebung

I.e.

$$\underline{\mathcal{L}^{-1}\left(F(s)\right) = H(t-2)\,\frac{2}{3}\sin(3t-6)}$$

Aufgabe: Man finde $\mathcal{L}^{-1}(\ldots)$ von: (i) $F(s) = \dfrac{2s+6}{s^2+4}$

(ii) $F(s) = \dfrac{1}{s^2(s^2+16)}$

Lösg.: (i)

$$\mathcal{L}^{-1}\left(\frac{2s+6}{s^2+4}\right) = 2\,\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + 6\,\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right)$$

$$= 2\cos(2t) + 3\sin(2t)$$

$$\left.\begin{tikzcd} \cos(t) \arrow[r, "\mathcal{L}"] \arrow[d] & \frac{s}{s^2+1} \arrow[d] \\ \cos(2t) \arrow[r, "\mathcal{L}"] & \frac{s}{s^2+4} = \frac{\frac{s}{4}}{\left(\frac{s}{2}\right)^2+1} \end{tikzcd}\right\} \text{Ähnlichkeit}$$

$$\left.\begin{tikzcd} \frac{1}{2}\sin(t) \arrow[r, "\mathcal{L}"] \arrow[d] & \frac{\frac{1}{2}}{s^2+1} \arrow[d] \\ \frac{1}{2}\sin(2t) \arrow[r, "\mathcal{L}"] & \frac{1}{s^2+4} = \frac{\frac{1}{4}}{\left(\frac{s}{2}\right)^2+1} \end{tikzcd}\right\} \text{Ähnlichkeit}$$

(ii)

$$\frac{1}{s^2(s^2+16)} \underset{\text{PBZ}}{=} \frac{1}{16s^2} - \frac{1}{16(s^2+16)}$$

$$\longrightarrow \mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+16)}\right) = \frac{1}{16}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{16}\mathcal{L}^{-1}\left(\frac{1}{s^2+16}\right)$$

$$= \frac{1}{16}t - \frac{1}{64}\sin(4t)$$

$$\left.\begin{tikzcd} \frac{1}{4}\sin(t) \arrow[r, "\mathcal{L}"] \arrow[d] & \frac{\frac{1}{4}}{s^2+1} \arrow[d] \\ \frac{1}{4}\sin(4t) \arrow[r, "\mathcal{L}"] & \frac{1}{s^2+16} = \frac{\frac{1}{16}}{\left(\frac{s}{4}\right)^2+1} \end{tikzcd}\right\} \text{Ähnlichkeit}$$

Beispiele von DGL

(i)

$$\boxed{\begin{aligned} y'' + y &= 0 \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned}}$$

$$\mathcal{L}(y) = Y$$

$$\mathcal{L}(y') = sY - y(0)$$

$$\mathcal{L}(y'') = s(sY - y(0)) - y'(0)$$

$$= s^2 Y - s\,y(0) - y'(0)$$

$$= s^2 Y - s$$

$\longrightarrow$ DGL wird zu: $\boxed{s^2 Y - s + Y = 0}$

i.e. $Y(s^2+1) = s$

$$Y(s) = \frac{s}{s^2+1}$$

$$\longrightarrow \underline{y(t)} = \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \underline{\cos(t)}$$

(ii)

$$\boxed{\begin{aligned} y'' - 6y' + 9y &= t^2 e^{3t} \\ y(0) &= 2 \\ y'(0) &= 6 \end{aligned}}$$

$$\mathcal{L}(y) = Y$$

$$\mathcal{L}(y') = sY - y(0) = sY - 2$$

$$\mathcal{L}(y'') = s(sY - 2) - y'(0)$$

$$= s^2 Y - 2s - 6$$

$$\mathcal{L}(t^2 e^{3t}) = \frac{2}{(s-3)^3}$$

$$\begin{array}{ccc} t^2 & \xrightarrow{\mathcal{L}} & \frac{2}{s^3} \\ \downarrow \cdot e^{3t} & & \downarrow s \mapsto s-3 \\ t^2 e^{3t} & \xrightarrow{\mathcal{L}} & \frac{2}{(s-3)^3} \end{array}$$

$\longrightarrow$ DGL wird zu: $\boxed{s^2 Y - 2s - 6 - 6sY + 12 + 9Y = \frac{2}{(s-3)^3}}$

i.e. $Y(s^2 - 6s + 9) = \frac{2}{(s-3)^3} + 2s - 6$

i.e. $Y = \frac{2}{(s-3)^3 \underbrace{(s^2 - 6s + 9)}_{=(s-3)^2}} + \frac{2s-6}{s^2 - 6s + 9}$

$$= \frac{2}{(s-3)^5} + \frac{2}{s-3}$$

$\mathcal{L}^{-1}\left(\frac{2}{s-3}\right) = 2e^{3t}$

$\mathcal{L}^{-1}\left(\frac{2}{(s-3)^5}\right) = \frac{1}{4!} e^{3t} t^4$

$$\left.\begin{array}{ccc} \frac{1}{4!} t^4 & \xrightarrow{\mathcal{L}} & \frac{1}{s^5} \\ \downarrow \cdot e^{3t} & & \downarrow \\ \frac{1}{4!} e^{3t} t^4 & \xrightarrow{\mathcal{L}} & \frac{1}{(s-3)^5} \end{array}\right\} \text{Dämpfung}$$

$\longrightarrow$ $\underline{y(t) =}$ $\mathcal{L}^{-1}\left(\frac{2}{(s-3)^5}\right) + \mathcal{L}^{-1}\left(\frac{2}{s-3}\right)$

$$= \frac{2}{4!} e^{3t} t^4 + 2e^{3t} = \underline{\left(\frac{2}{4!} t^4 + 2\right) e^{3t}}$$

<u>Aufgabe</u>: Man löse mit Laplace:

(i) $\left.\begin{array}{r} y'' + 2y' + y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{array}\right\}$

(ii) $\left.\begin{array}{r} y' + 2y = 4t \\ y(0) = 1 \end{array}\right\}$