Wiederholung

Kommutative Diagramme

Bsp: **Dämpfung**:

$$\begin{tikzcd} f(t) \arrow[r, "\mathcal{L}"] \arrow[d, "\cdot e^{-at}"'] & F(s) \arrow[d, "s \mapsto s+a"] \\ e^{-at} f(t) \arrow[r, "\mathcal{L}"] & F(s+a) \end{tikzcd}$$

Zeitverschiebung

$$\begin{tikzcd} H(t) f(t) \arrow[r, "\mathcal{L}"] \arrow[d, "t \mapsto t-t_0"'] & F(s) \arrow[d, "\cdot e^{-t_0 s}"] \\ H(t-t_0) f(t-t_0) \arrow[r, "\mathcal{L}"] & e^{-t_0 s} F(s) \end{tikzcd}$$

Ähnlichkeit:

$$\begin{tikzcd} f(t) \arrow[r, "\mathcal{L}"] \arrow[d, "t \mapsto at"'] & F(s) \arrow[d] \\ f(at) \arrow[r, "\mathcal{L}"] & \frac{1}{a} F\left(\frac{s}{a}\right) \end{tikzcd}$$

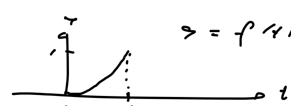
Diagramme kombinieren:

$$\begin{tikzcd} \sin(t) \arrow[r, "\mathcal{L}"] \arrow[d, "t \mapsto \omega t"'] & \frac{1}{s^2+1} \arrow[d] \\ \sin(\omega t) \arrow[r, "\mathcal{L}"] \arrow[d, "\cdot e^{-3t}"'] & \frac{1}{\omega} \frac{1}{\left(\frac{s}{\omega}\right)^2+1} \arrow[d] \\ e^{-3t} \sin(\omega t) \arrow[r, "\mathcal{L}"] & \frac{1}{\omega} \frac{1}{\left(\frac{s+3}{\omega}\right)^2+1} \end{tikzcd}$$

Ähnlichkeit (erster Schritt), Dämpfung (zweiter Schritt)

Stückweise def. Fkt.

Bsp: $f(t) = \begin{cases} 0 & : \ t < 0 \\ t^2 & : \ 0 \leq t < 1 \\ 0 & : \ t \geq 1 \end{cases}$



$$f(t) = \left(H(t) - H(t-1)\right) t^2 = H(t)\, t^2 - H(t-1)\, t^2$$

$$H(t)\,t^2 \xrightarrow{\mathcal{L}} \frac{2}{s^3}$$

$$\begin{tikzcd} H(t)(t+1)^2 = H(t)\left(t^2+2t+1\right) \arrow[r, "\mathcal{L}"] \arrow[d, "t \mapsto t-1"'] & \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} \arrow[d] \\ H(t-1)\, t^2 \arrow[r, "\mathcal{L}"] & e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) \end{tikzcd}$$

I.e. $\underline{\mathcal{L}(f(t)) = \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)}$

Aufgabe: Sei $f(t) = \begin{cases} \cos(t) & : 0 \le t < 2\pi \\ 0 & : t \ge 2\pi \end{cases}$

Ges: $\mathcal{L}(f(t))$

Lösg:

$$f(t) = \big(H(t) - H(t-2\pi)\big)\cos(t)$$

$$= H(t)\cos(t) - H(t-2\pi)\cos(t)$$

$$H(t)\cos(t) \xrightarrow{\mathcal{L}} \frac{s}{s^2+1}$$

$$H(t)\cos(t+2\pi) = H(t)\cos(t) \xrightarrow{\mathcal{L}} \frac{s}{s^2+1}$$

$$t \mapsto t-2\pi$$

$$H(t-2\pi)\cos(t) \xrightarrow{\mathcal{L}} e^{-2\pi s}\frac{s}{s^2+1}$$

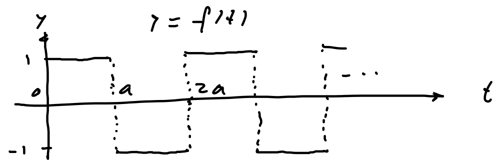
$$\longrightarrow \underline{\mathcal{L}(f(t)) = \frac{s}{s^2+1}\left(1 - e^{-2\pi s}\right)}$$

Periodische Fkt.

$$f : [0,\infty) \longrightarrow \mathbb{R}$$
$$t \longmapsto f(t) \qquad \text{mit: } f(t+T) = f(t)$$

$$\boxed{\mathcal{L}(f(t)) = \frac{1}{1-e^{-sT}} \int_0^T f(t)e^{-st}\,dt}$$

Bsp:



$$\mathcal{L}(f(t)) = \frac{1}{1-e^{-2as}}\left(\int_0^a 1 \cdot e^{-st}\,dt + \int_a^{2a} (-1)\cdot e^{-st}\,dt\right)$$

$$= \frac{1}{1-e^{-2as}}\left(-\frac{1}{s}e^{-st}\Big|_0^a + \frac{1}{s}e^{-st}\Big|_a^{2a}\right)$$

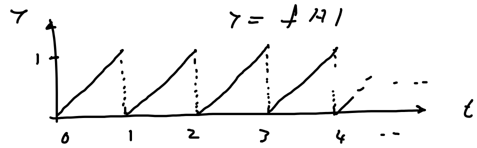
$$= \frac{1}{s(1-e^{-2as})}\left(-e^{-sa} + 1 + e^{-s2a} - e^{-sa}\right)$$

$$= \frac{1}{s(1-e^{-2as})}\underbrace{\left(1 - 2e^{-sa} + e^{-2sa}\right)}_{(1-e^{-sa})^2}$$

$$= \frac{(1-e^{-sa})^{\cancel{2}}}{s(1+e^{-sa})\cancel{(1-e^{-sa})}}$$

$$= \underline{\underline{\frac{1-e^{-sa}}{s(1+e^{-sa})}}}$$

Aufgabe:



Ges: $\mathcal{L}(f(t))$

Lösg:

$$\mathcal{L}(f(t)) = \frac{1}{1-e^{-sT}} \int_0^T f(t) e^{-st}\,dt$$

$$= \frac{1}{1-e^{-s}} \int_0^1 t\, e^{-st}\,dt$$

$\int f g' = fg - \int f' g$

$$= \frac{1}{1-e^{-s}} \left(t\left(-\frac{1}{s} e^{-st}\right)\Big|_0^1 - \int_0^1 -\frac{1}{s} e^{-st}\,dt \right)$$

$$= \frac{1}{1-e^{-s}} \left(-\frac{1}{s} e^{-s} - \frac{1}{s^2}\left(e^{-s} - 1\right)\right)$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})}$$