Darstellung der Eigenschaften in kommutativen Diagrammen

(i) Ähnlichkeit $\mathcal{L}(f(at)) = \frac{1}{a} F\left(\frac{s}{a}\right)$

$$\begin{tikzcd} f(t) \arrow[r, "\mathcal{L}"] \arrow[d, "t \mapsto at"'] & F(s) \arrow[d] \\ f(at) \arrow[r, "\mathcal{L}"] & \frac{1}{a} F\left(\frac{s}{a}\right) \end{tikzcd}$$

Bsp:

$$\begin{tikzcd} \cos(t) \arrow[r, "\mathcal{L}"] \arrow[d, "t \mapsto \omega t"'] & \frac{s}{s^2+1} \arrow[d] \\ \cos(\omega t) \arrow[r, "\mathcal{L}"] & \frac{1}{\omega} \cdot \frac{\frac{s}{\omega}}{\left(\frac{s}{\omega}\right)^2+1} \end{tikzcd}$$

(ii) Dämpfung $\mathcal{L}(e^{-at} f(t)) = F(s+a)$

$$\begin{tikzcd} f(t) \arrow[r, "\mathcal{L}"] \arrow[d, "\cdot e^{-at}"'] & F(s) \arrow[d, "s \mapsto s+a"] \\ e^{-at} f(t) \arrow[r, "\mathcal{L}"] & F(s+a) \end{tikzcd}$$

Bsp:

$$\begin{tikzcd} \sin(t) \arrow[r, "\mathcal{L}"] \arrow[d, "\cdot e^{-3t}"'] & \frac{1}{s^2+1} \arrow[d] \\ e^{-3t} \sin(t) \arrow[r, "\mathcal{L}"] & \frac{1}{(s+3)^2+1} \end{tikzcd}$$

Können Diagramme kombinieren:

Bsp: $\mathcal{L}(e^{-3t} \sin(\omega t)) = ?$

$$\begin{tikzcd} \sin(t) \arrow[r, "\mathcal{L}"] \arrow[d, "t \mapsto \omega t"'] & \frac{1}{s^2+1} \arrow[d] \\ \sin(\omega t) \arrow[r, "\mathcal{L}"] \arrow[d, "\cdot e^{-3t}"'] & \frac{1}{\omega} \cdot \frac{1}{\left(\frac{s}{\omega}\right)^2+1} \arrow[d] \\ e^{-3t} \sin(\omega t) \arrow[r, "\mathcal{L}"] & \frac{1}{\omega} \cdot \frac{1}{\left(\frac{s+3}{\omega}\right)^2+1} \end{tikzcd}$$

I.e. $\mathcal{L}(e^{-3t} \sin(\omega t)) = \frac{1}{\omega} \cdot \frac{1}{\left(\frac{s+3}{\omega}\right)^2+1} = \frac{\omega}{(s+3)^2+\omega^2}$

(iii) Zeitverschiebung:

$\mathcal{L}(H(t-t_0) f(t-t_0)) = e^{-t_0 s} F(s)$

$$\begin{tikzcd} H(t) f(t) \arrow[r, "\mathcal{L}"] \arrow[d, "t \mapsto t-t_0"'] & F(s) \\ H(t-t_0) f(t-t_0) \arrow[r] & e^{-t_0 s} F(s) \end{tikzcd}$$

(iv) Ableitung Zeitbereich

$\mathcal{L}(f'(t)) = s F(s) - f(0)$

$$\begin{tikzcd} f(t) \arrow[r, "\mathcal{L}"] \arrow[d, "\frac{d}{dt}"'] & F(s) \arrow[d] \\ f'(t) \arrow[r, "\mathcal{L}"] & s F(s) - f(0) \end{tikzcd}$$

Bsp:

$$\begin{tikzcd}
\boxed{f(t)} \arrow[r, "\mathcal{L}"] \arrow[d, "\frac{d}{dt}"'] & F(s) \arrow[d] \\
\boxed{f'(t)} \arrow[r, "\mathcal{L}"] \arrow[d, "\frac{d}{dt}"'] & sF(s) - \boxed{f(0)} \arrow[d] \\
\boxed{f''(t)} \arrow[r, "\mathcal{L}"] \arrow[d, "\frac{d}{dt}"'] & s\left(sF(s) - f(0)\right) - \boxed{f'(0)} \arrow[d] \\
f'''(t) \arrow[r, "\mathcal{L}"] & s\Big(s(sF(s) - f(0)) - f'(0)\Big) - \boxed{f''(0)}
\end{tikzcd}$$

(v) Integral Zeitbereich

$$\mathcal{L}\left(\int_0^t f(u)\,du\right) = \frac{F(s)}{s}$$

$$\boxed{\begin{tikzcd}
f(t) \arrow[r, "\mathcal{L}"] \arrow[d, "\int \ldots"'] & F(s) \arrow[d, "\cdot \frac{1}{s}"] \\
\int_0^t f(u)\,du \arrow[r] & \frac{F(s)}{s}
\end{tikzcd}}$$

(vi) Ableitung Bildbereich

$$\frac{dF}{ds}(s) = \mathcal{L}\left(-t\, f(t)\right)$$

$$\boxed{\begin{tikzcd}
f(t) \arrow[r, "\mathcal{L}"] \arrow[d, "\cdot(-t)"'] & F(s) \arrow[d, "\frac{d}{ds}"] \\
-t\, f(t) \arrow[r, "\mathcal{L}"] & \frac{dF}{ds}(s)
\end{tikzcd}}$$

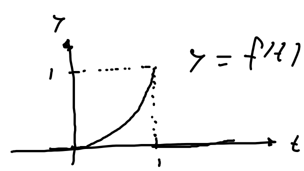
Bsp:

$$\begin{tikzcd}
\sinh(t) \arrow[r, "\mathcal{L}"] \arrow[d, "\cdot t"'] & \frac{1}{s^2-1} \arrow[d, "-\frac{d}{ds}"] \\
t \sinh(t) \arrow[r, "\mathcal{L}"] & \frac{2s}{(s^2-1)^2}
\end{tikzcd}$$

Stückweise definierte Fkt.:

Illustration an Bsp:

$$f(t) = \begin{cases} 0 & : \ t < 0 \\ t^2 & : \ 0 \le t < 1 \\ 0 & : \ t \geq 1 \end{cases}$$



$f(t)$ geschrieben mit Fensterfkt.:

$$f(t) = \left(H(t) - H(t-1)\right) t^2$$

$$= H(t)\, t^2 - H(t-1)\, t^2$$

Bestimmen $\mathcal{L}(f(t))$: 1. Term: $\mathcal{L}(H(t)\, t^2) = \mathcal{L}(t^2) = \frac{2}{s^3}$

2. Term: Mit Diagramm:

$$\begin{tikzcd}
H(t)(t+1)^2 = H(t)(t^2+2t+1) \arrow[r, "\mathcal{L}"] \arrow[d, "t \mapsto t-1"'] & \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} \arrow[d, "\cdot e^{-s}"] \\
H(t-1)\, t^2 \arrow[r, "\mathcal{L}"] & e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)
\end{tikzcd}$$

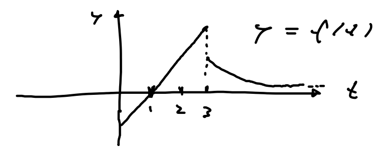
$$\longrightarrow \underline{\mathcal{L}(f(t)) = \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)}$$

Aufgabe:

$$f(t) = \begin{cases} 0 & : t < 0 \\ t-1 & : 0 \le t < 3 \\ 5e^{-\frac{t}{2}} & : t \ge 3 \end{cases}$$

Man skizziere $f(t)$ & schreibe $f(t)$ in einer Zeile. Hinweis: $5e^{-\frac{3}{2}} = 1,1\ldots$

Man finde $\mathcal{L}(f(t))$.



Lösg:

$$f(t) = \big(H(t) - H(t-3)\big)(t-1) + H(t-3)5e^{-\frac{t}{2}}$$

$$= H(t)(t-1) + H(t-3)\left(-t+1+5e^{-\frac{t}{2}}\right)$$

Trafo: 1. Term: $\mathcal{L}\big(H(t)(t-1)\big) = \mathcal{L}(t) - \mathcal{L}(1)$

$$= \frac{1}{s^2} - \frac{1}{s}$$

2. Term:

$$H(t)\Big(\underbrace{-(t+3)+1}_{-t-2}+\overbrace{5e^{-\frac{t+3}{2}}}^{=5e^{-\frac{3}{2}}e^{-\frac{t}{2}}}\Big) \xrightarrow{\mathcal{L}} -\frac{1}{s^2} - \frac{2}{s} + 5e^{-\frac{3}{2}} \cdot \frac{1}{s+\frac{1}{2}}$$

$$\Big\downarrow t \mapsto t-3 \qquad\qquad\qquad\qquad \Big\downarrow \cdot e^{-3s}$$

$$H(t-3)\left(-t+1+5e^{-\frac{t}{2}}\right) \xrightarrow{\mathcal{L}} \left(-\frac{1}{s^2} - \frac{2}{s} + 5e^{-\frac{3}{2}}\frac{1}{s+\frac{1}{2}}\right)e^{-3s}$$

$$\longrightarrow \underline{\mathcal{L}(f(t)) = \frac{1}{s^2} - \frac{1}{s} + e^{-3s}\left(-\frac{1}{s^2} - \frac{2}{s} + \frac{5e^{-\frac{3}{2}}}{s+\frac{1}{2}}\right)}$$

Periodische Funktionen

$$f : [0, \infty) \longrightarrow \mathbb{R}$$
$$t \longmapsto f(t)$$ sei periodisch mit Periode $T > 0$,

i.e. $f(t+T) = f(t)$

Es gilt:

$$\boxed{\mathcal{L}(f(t)) = \frac{1}{1-e^{-sT}} \int_0^T f(t)e^{-st}\,dt}$$

Herleitung: Geometrische Reihe:

$$\sum_{n=0}^{N} x^n = 1 + x + x^2 + x^3 + \cdots + x^N \qquad (1)$$

$$x\sum_{n=0}^{N} x^n = x + x^2 + x^3 + \cdots + x^{N+1} \qquad (2)$$

(1) - (2):

$$(1-x)\sum_{n=0}^{N} x^n = 1 - x^{N+1} \quad / : (1-x)$$

$$\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$$ Mit $x < 1$: $x^{N+1} \xrightarrow{N \to \infty} 0$

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$$

<u>Beweis:</u>

$$\mathcal{L}(f(t)) = \int_0^\infty f(t) e^{-st}\, dt$$

$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} f(t) e^{-st}\, dt$$

$$\underset{\substack{\uparrow \\ u = t - nT}}{=} \sum_{n=0}^{\infty} \int_0^T f(u+nT)\, e^{-s(u+nT)}\, du$$

$$\underset{\substack{\uparrow \\ f \text{ per.}}}{=} \sum_{n=0}^{\infty} \int_0^T f(u)\, e^{-s(u+nT)}\, du$$

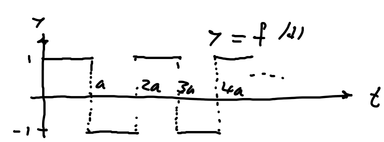
$$= \sum_{n=0}^{\infty} e^{-snT} \int_0^T f(u) e^{-su}\, du$$

$$= \left(\int_0^T f(u)\, e^{-su} du \right) \underbrace{\sum_{n=0}^{\infty} \left(e^{-sT}\right)^n}_{\underset{\substack{\uparrow \\ \text{geom. Reihe}}}{=} \frac{1}{1-e^{-sT}}}$$

$$= \underline{\frac{1}{1-e^{-sT}} \int_0^T f(u)\, e^{-su}\, du} \qquad \square$$

<u>Bsp:</u>

Sei $f(t) = \begin{cases} 1 & : 0 \leq t < a \\ -1 & : a \leq t < 2a. \end{cases}$



und sei $f(t+2a) = f(t)$.

$$\mathcal{L}(f(t)) = \frac{1}{1-e^{-Ts}} \int_0^T f(t) e^{-st}\, dt$$

$$= \frac{1}{1-e^{-2as}} \left(\int_0^a 1 \cdot e^{-st} dt + \int_a^{2a} (-1) \cdot e^{-st} dt \right)$$

$$= \ldots = \underline{\frac{1-e^{-as}}{s(1+e^{-as})}}$$