Laplace - Trafo

Wiederholung: Def: $\mathcal{L}(f(t)) = \underset{\substack{\uparrow \\ \text{Bildfkt.}}}{F(s)} = \int_0^\infty \underset{\substack{\uparrow \\ \text{Originalfkt.}}}{f(t)} e^{-st} \, dt$

Trafos: $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$

$\mathcal{L}(\cos(t)) = \frac{s}{s^2+1}$; $\mathcal{L}(\sin(t)) = \frac{1}{s^2+1}$

$\mathcal{L}(e^{wt}) = \frac{1}{s-w}$

Eigenschaften: Linearität: $\mathcal{L}(C_1 f_1(t) + C_2 f_2(t))$

$= C_1 \mathcal{L}(f_1(t)) + C_2 \mathcal{L}(f_2(t))$.

Ähnlichkeit: $\mathcal{L}(f(at)) = \frac{1}{a} F(\frac{s}{a})$, $a > 0$

mit $F(s) = \mathcal{L}(f(t))$

Dämpfung: $\mathcal{L}(e^{-at} f(t)) = F(s+a)$, $a > 0$

mit $F(s) = \mathcal{L}(f(t))$

Zeitverschiebung rechts:

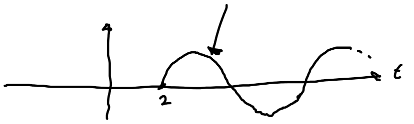
$\mathcal{L}(H(t-a) f(t-a)) = e^{-as} F(s)$, $a > 0$

mit $F(s) = \mathcal{L}(f(t))$

und $H(t-a)$: Heaviside - Fkt.

$H(t) = \begin{cases} 0 : t < 0 \\ 1 : t \geq 0 \end{cases}$

Bsp: $H(t-2)\sin(t-2)$



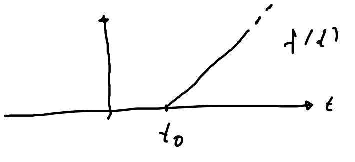
$\mathcal{L}(H(t-2)\sin(t-2)) = e^{-2s} \underbrace{F(s)}_{= \mathcal{L}(\sin(t))}$

$= \frac{e^{-2s}}{s^2+1}$

Aufgabe: Man bestimme $\mathcal{L}(f(t))$, wobei

$f(t) = \begin{cases} 0 & : t < t_0 \\ t - t_0 & : t \geq t_0 \end{cases}$, $t_0 > 0$

Lösg.:



I.e. $f(t) = H(t-t_0)(t-t_0)$

$\longrightarrow \mathcal{L}(f(t)) = \mathcal{L}(H(t-t_0)(t-t_0))$

$= e^{-t_0 s} \underbrace{\frac{1}{s^2}}_{= \mathcal{L}(t)}$

Ableitung Zeitbereich: Beweis:

$$\mathcal{L}(f'(t)) = \int_0^\infty f'(t)\, e^{-st}\, dt$$

$\int f' g = f g - \int f g'$ (part. Int.)

$$= \underbrace{f(t)\, e^{-st}\Big|_0^\infty}_{-f(0)} - \int_0^\infty f(t)(-s)\, e^{-st}\, dt$$

$$= -f(0) + s \underbrace{\int_0^\infty f(t)\, e^{-st}\, dt}_{= \mathcal{L}(f(t)) = F(s)}$$

$$= s F(s) - f(0) \qquad \square$$

I.e.

$$\boxed{\begin{array}{l} \mathcal{L}(f'(t)) = s F(s) - f(0) \\ \qquad \text{mit } F(s) = \mathcal{L}(f(t)) \end{array}}$$

2-te Ableitung:

$$\boxed{\mathcal{L}(f''(t))} = \mathcal{L}\left(\frac{d}{dt} \left(\frac{df}{dt} \right)(t) \right)$$

$$= s\, \mathcal{L}\left(\frac{df}{dt}(t) \right) - \frac{df}{dt}(0)$$

$$= s \Big(s \underset{\mathcal{L}(f(t))}{\underset{\shortparallel}{F(s)}} - f(0) \Big) - \frac{df}{dt}(0)$$

$$= \boxed{s^2 F(s) - s f(0) - f'(0)}$$

Allgemein:

$$\boxed{\begin{array}{l} \mathcal{L}(f^{(n)}(t)) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots \\ \qquad \qquad \qquad \dots - f^{(n-1)}(0) \end{array}}$$

Bsp: $f(t) = \sin(t) \longrightarrow f(0) = 0$

$f'(t) = \cos(t)$

$$\longrightarrow \mathcal{L}(\cos(t)) = \mathcal{L}(f'(t)) = s F(s) - f(0)$$

$$= s\, \mathcal{L}(\sin(t)) - \underset{0}{\underset{\shortparallel}{f(0)}}$$

$$= \underline{\frac{s}{s^2+1}}$$

Ableitung Bildbereich

Sei $F(s) = \mathcal{L}(f(t)) \longrightarrow \boxed{\frac{dF}{ds}(s) = \mathcal{L}(-t f(t))}$

Beweis:

$$\frac{dF}{ds}(s) = \frac{d}{ds}\left(\int_0^\infty f(t)e^{-st}\,dt\right)$$

$$= \int_0^\infty f(t)\,\frac{\partial}{\partial s}\left(e^{-st}\right)dt$$

$$= \int_0^\infty f(t)\left(-te^{-st}\right)dt = \int_0^\infty -t\,f(t)\,e^{-st}\,dt$$

$$= \mathcal{L}(-t f(t)) \qquad \square$$

Allgemein:

$$\boxed{\frac{d^nF}{ds^n}(s) = \mathcal{L}\left((-t)^n f(t)\right)}$$

Bsp:

$$\mathcal{L}(t\sin(t)) = -\frac{d}{ds}\mathcal{L}(\sin(t))$$

$$= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

Analog: $\mathcal{L}(t\cos(t)) = \dots = \dfrac{s^2-1}{(s^2+1)^2}$

Integral Zeitbereich:

$$\boxed{\begin{aligned}\mathcal{L}\left(\int_0^t f(u)\,du\right) &= \frac{F(s)}{s}\\ \text{wobei } F(s) &= \mathcal{L}(f(t))\end{aligned}}$$

Beweis:

$$\mathcal{L}\left(\int_0^t f(u)\,du\right) = \int_0^\infty \left(\int_0^t f(u)\,du\right)e^{-st}\,dt$$

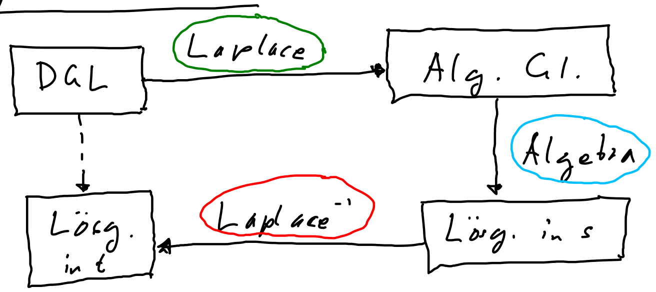
$$\int F G' = FG - \int F'G \quad \left(\begin{matrix}\text{part.}\\ \text{Int.}\end{matrix}\right)$$

$$= \underbrace{\int_0^t f(u)\,du\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty}_{0} - \int_0^\infty f(t)\left(-\frac{1}{s}e^{-st}\right)dt$$

$$= \frac{1}{s}\int_0^\infty f(t)\,e^{-st}\,dt = \frac{1}{s}F(s) \qquad \square$$

Einschub: **Lösung Anfangswertproblem:**

Idee:



Bsp: (:)

$$\left.\begin{aligned} f''(t) &= -f(t)\\ f(0) &= 0\\ f'(0) &= 1\end{aligned}\right\} \quad \underline{\text{Anfangswertproblem}}$$

Laplace - Trafo:

$$\underbrace{s^2 F(s) - s f(0) - f'(0)}_{\mathcal{L}(f''(t))} = \underbrace{-F(s)}_{= \mathcal{L}(-f(t))}$$

$$\longrightarrow F(s)(s^2+1) = s\underbrace{f(0)}_{=0} + \underbrace{f'(0)}_{=1}$$

$$\longrightarrow F(s) = \frac{1}{s^2+1}$$

$$\text{Laplace}^{-1} \longrightarrow f(t) = \sin(t)$$

(ii) $\left.\begin{array}{l} f'(t) + 3f(t) = e^{-t} \\ f(0) = 0 \end{array}\right\}$

$$\text{Laplace} \longrightarrow \underbrace{sF(s) - \overset{\overset{0}{\parallel}}{f(0)}}_{=\mathcal{L}(f'(t))} + \underbrace{3F(s)}_{=\mathcal{L}(3f(t))} = \underbrace{\frac{1}{s+1}}_{=\mathcal{L}(e^{-t})}$$

$$\text{Algebra} \longrightarrow F(s)(s+3) = \frac{1}{s+1}$$

$$\longrightarrow F(s) = \frac{1}{(s+1)(s+3)} \underset{\underset{\text{PBZ}}{\uparrow}}{=} \frac{1}{2(s+1)} - \frac{1}{2(s+3)}$$

$$\text{Laplace}^{-1} \longrightarrow \text{Wir wissen: } \mathcal{L}(e^{-t}) = \frac{1}{s+1} \; ; \; \mathcal{L}(e^{-3t}) = \frac{1}{s+3}$$

$$\longrightarrow f(t) = \frac{1}{2} e^{-t} - \frac{1}{2} e^{-3t}$$

(Wir sehen: Problematik: Trafo^{-1} !)

Darstellung der Eigenschaften in Kommutativen Diagrammen

(i) Ähnlichkeit: $\mathcal{L}(f(at)) = \frac{1}{a} F\left(\frac{s}{a}\right)$

Diagramm:

$$\begin{tikzcd}
f(t) \arrow[r, "\mathcal{L}"] \arrow[d, "t \mapsto at"'] & F(s) \arrow[d] \\
f(at) \arrow[r, "\mathcal{L}"] & \frac{1}{a} F\left(\frac{s}{a}\right)
\end{tikzcd}$$

Bsp:

$$\begin{tikzcd}
\cos(t) \arrow[r, "\mathcal{L}"] \arrow[d, "t \mapsto \omega t"'] & \frac{s}{s^2+1} \arrow[d] \\
\cos(\omega t) \arrow[r, "\mathcal{L}"] & \frac{1}{\omega} \frac{\frac{s}{\omega}}{\left(\frac{s}{\omega}\right)^2+1}
\end{tikzcd}$$

(ii) Dämpfung $\mathcal{L}(e^{-at}f(t)) = F(s+a)$

Diagramm:

$$\begin{tikzcd}
f(t) \arrow[r, "\mathcal{L}"] \arrow[d, "\cdot e^{-at}"'] & F(s) \arrow[d, "s \mapsto s+a"] \\
e^{-at}f(t) \arrow[r, "\mathcal{L}"] & F(s+a)
\end{tikzcd}$$

Bsp:

$$\begin{tikzcd}
\sin(t) \arrow[r, "\mathcal{L}"] \arrow[d, "\cdot e^{-3t}"'] & \frac{1}{s^2+1} \arrow[d] \\
e^{-3t}\sin(t) \arrow[r, "\mathcal{L}"] & \frac{1}{(s+3)^2+1}
\end{tikzcd}$$