# VECTOR ANALYSIS
# PROBLEM SETS

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## 1. Hand in: 28.09.20, count: 10Pts.

1.1. **Circle rolling on sine-graph.** A circle with radius $r$ rolls on top of the graph of the sine function. Find a parametric representation of the path traced out by the center of the circle. Assume that the radius $r$ is sufficiently small, such that the circle travels through the valleys of the sine graph. Hint: Any graph $y = f(x)$ can be written as a parametric curve by using $x$ as the parameter: $\overrightarrow{r}(x) = (x, f(x))$.

1.2. **Charged particle in homogeneous magnetic field.** The trajectory of a charged particle travelling through a magnetic field $\overrightarrow{B} = (0, 0, B)$ (with $B$ a constant), is given by

$$\overrightarrow{r}(t) = \begin{pmatrix} R\cos(\omega t) \\ R\sin(\omega t) \\ ct \end{pmatrix},$$

where $R$, $\omega$ and $c$ are constants.

(i) Compute the length of the trajectory traced out between $t = 0$ and $t = 5T$, where $T = 2\pi/\omega$.
(ii) Show that the given trajectory $\overrightarrow{r}(t)$ is in agreement with Newton's law of motion $\overrightarrow{F} = m\overrightarrow{a}$, where the acting force $\overrightarrow{F}$ is given by $\overrightarrow{F} = q\overrightarrow{v} \times \overrightarrow{B}$, i.e. given by the Lorentz force.
(iii) Find an expression for $\omega$ in terms of $q$, $B$, $m$.

## 2. Hand in: 05.10.20, count: 10Pts.

2.1. **Arc length of graph.** We consider a curve given by the graph of a function $f : [a, b] \to \mathbb{R}$, $x \mapsto y = f(x)$. Use the arc length formula for curves, as given in class, to show that the arc length of the graph is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2}dx.$$

2.2. **Gravitational force.** The gravitational potential outside earth is given in good approximation by

$$\phi(x, y, z) = -\frac{GM}{r(x, y, z)}, \qquad \text{where} \qquad r(x, y, z) = \sqrt{x^2 + y^2 + z^2}.$$

Here $G$ is the gravitational constant, $M$ is the mass of the earth and the coordinates have been chosen such that the center of the earth coincides with the origin. The gravitational force acting on a body of mass $m$ is given by $\overrightarrow{F}_g = m\overrightarrow{g}$, where $\overrightarrow{g} = -\overrightarrow{\nabla}\phi$ is the gravitational field. Show that the gravitational force $\overrightarrow{F}_g$ points towards the origin and its magnitude is given by

$$\left|\overrightarrow{F}_g\right| = \frac{GMm}{r^2}.$$

## 3. Hand in: 12.10.20, count: 10Pts.

3.1. **Work for bringing object into orbit.** An object with mass $m$ is brought from the surface of the earth to a height $h$ by following the trajectory given by

$$\begin{aligned} \overrightarrow{r} &: [0,1] \to \mathbb{R}^3 \\ t &\mapsto \overrightarrow{r}(t) = (r_E + th) \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \\ 0 \end{pmatrix}. \end{aligned}$$

Here $r_E$ is the radius of the earth.

(i) Compute the necessary work using

$$W = \int_C \overrightarrow{F}_g \cdot \overrightarrow{dr},$$

where $\overrightarrow{F}_g$ is the gravitational force as given in the previous exercise.

(ii) Compare the result with the one that would be obtained on a purely radial trajectory.

(iii) Explain why the results have to be the same.

3.2. **Path independedece and closed loop.** Let $\overrightarrow{F}$ be a vectorfield defined over a domain $\Omega$. Show that the following statementes are equivalent:

(i) $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ is path independent,

(ii) $\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = 0$ for any closed loop $C$.

Hint: Use $\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = -\int_{C^-} \overrightarrow{F} \cdot d\overrightarrow{r}$, where $C$ and $C^-$ possess the same trace but opposite orientation.