Partielle Integration ["Umkehrung Produktregel"]

$\boxed{\text{Repetition}}$

$$(fg)' = f'g + fg' \xrightarrow{\int \dots} \underbrace{\int (fg)'}_{= fg} = \int f'g + \int fg'$$

$$\rightarrow \int fg' = fg - \int f'g$$

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)dx$$

Bsp.:

$$\int \underbrace{4x}_{\int f} \underbrace{\cos(2-3x)}_{g'} dx = \frac{4x\sin(2-3x)}{-3} - \int \frac{4\sin(2-3x)}{-3} dx$$

$$= f g - \int f' g$$

$$= -\frac{4}{3} x \sin(2-3x) + \frac{4}{3} \int \sin(2-3x) dx$$

$$= \text{———}" \text{———} + \frac{4}{3}\left(\frac{-\cos(2-3x)}{-3}\right) + C$$

$$= -\frac{4}{3} x \sin(2-3x) + \frac{4}{9}\cos(2-3x) + C$$

Bsp.:

$$\int \underbrace{(3t+t^2)}_{\int f} \underbrace{\sin(2t)}_{g'} dt = (3t+t^2)\left(-\frac{\cos(2t)}{2}\right) - \int (3+2t)\left(-\frac{\cos(2t)}{2}\right) dt$$

$$= f g - \int f' g$$

$$= \text{———}" \text{———} + \frac{1}{2} \int \underbrace{(3+2t)}_{\int f} \underbrace{\cos(2t) dt}_{g'} = fg - \int f'g$$

$$= \text{———}" \text{———} + \frac{1}{2}\left((3+2t)\frac{\sin(2t)}{2} - \int 2\frac{\sin(2t)}{2} dt\right)$$

$$= \text{———}" \text{———} + \frac{3+2t}{4}\sin(2t) - \frac{\cos(2t)}{4} + C$$

$$= (3t+t^2)\left(-\frac{\cos(2t)}{2}\right) + \frac{3+2t}{4}\sin(2t) - \frac{\cos(2t)}{4} + C$$

Substitution: ["Umkehrung Kettenregel"]

Einfaches Bsp.:

$$\int e^{2x} dx = \frac{1}{2} e^{2x} + C$$

Überlegung mit Kettenregel $\left(\frac{d}{dx} e^{2x} = \underset{\uparrow\ \text{Innere Abl.}}{2} e^{2x}\right)$

Mit Subs.:

$$\int e^{2x} dx \underset{u=2x}{=} \int e^u \frac{du}{2} = \frac{1}{2}\int e^u du = \frac{1}{2} e^u + C \underset{u=2x}{=} \frac{1}{2} e^{2x} + C$$

$$\frac{du}{dx} = 2 \longrightarrow dx = \frac{du}{2}$$

Bsp.:

$$\int_{v=1}^{v=2} \underbrace{12v}_{\text{inn. Abl. von}} (7+6v^2)^9 dv \underset{u=7+6v^2}{=} \int_{u=13}^{u=31} 12v\, u^9 \frac{du}{12v} = \int_{13}^{31} u^9 du = \frac{u^{10}}{10}\Big|_{13}^{31} + \cancel{C} \underset{u=7+6v^2}{=} \frac{(7+6v^2)^{10}}{10} + C$$

$$u = 7+6v^2 \longrightarrow \frac{du}{dv} = 12v \longrightarrow dv = \frac{du}{12v}$$

$$\frac{31^{10}}{10} - \frac{13^{10}}{10}$$

Orange: mit Grenzen

Grenzen: $v = 1 \longrightarrow u = 7 + 6 \cdot 1^2 = 13$

$v = 2 \longrightarrow u = 7 + 6 \cdot 2^2 = 31$

Integration mit Partialbruchzerlegung

Bsp.: $\displaystyle\int \frac{3x+11}{x^2 - x - 6}\, dx$

Echt gebrochen: ja

PBZ: Lin.-Fakt. für Nenner: $x^2 - x - 6 = (x+2)(x-3)$

$$\longrightarrow \frac{3x+11}{x^2-x-6} = \frac{3x+11}{(x-3)(x+2)}$$

Ansatz: $\displaystyle\frac{3x+11}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2} \quad \Big/ \cdot (x-3)(x+2)$

$$3x + 11 = A(x+2) + B(x-3)$$

$x = -2: \quad 5 = -5B \longrightarrow \underline{B = -1}$

$x = 3: \quad 20 = 5A \longrightarrow \underline{A = 4}$

$$\longrightarrow \underline{\int \frac{3x+11}{x^2-x-6}\,dx} = \int \frac{3x+11}{(x-3)(x+2)}\,dx = \int \frac{4}{x-3}\,dx - \int \frac{1}{x+2}\,dx$$

$$= \underline{4 \log(x-3) - \log(x+2) + C}$$

Aufgabe: $\displaystyle\int \frac{x^2+4}{3x^3 + 4x^2 - 4x}\, dx$ Finden sie Ansatz für PBZ

Lösg.: $3x^3 + 4x^2 - 4x = x(\underbrace{3x^2 + 4x - 4}_{=0})$

$x = 0$ 1st NS

$\longrightarrow x = \dfrac{-4 \pm \sqrt{16 + 48}}{6} = \dfrac{-4 \pm 8}{6}$

$\longrightarrow$ NS sind: $x_+ = \frac{2}{3}$

$x_- = -2$

i.e. $3x^3 + 4x^2 - 4x = 3x\left(x - \frac{2}{3}\right)(x+2)$

$$\longrightarrow \frac{x^2+4}{3x^3+4x^2-4x} = \frac{x^2+4}{3x\left(x-\frac{2}{3}\right)(x+2)} = \frac{A}{x} + \frac{B}{x-\frac{2}{3}} + \frac{C}{x+2} \quad \Big/ \cdots$$

Ansatz

$$x^2 + 4 = A\left(x - \tfrac{2}{3}\right)(x+2) + Bx(x+2) + Cx\left(x - \tfrac{2}{3}\right)$$

$x = \frac{2}{3}: \quad \frac{4}{5} + 4 = B \cdot \frac{2}{3} \cdot \left(\frac{2}{3} + 2\right) \longrightarrow B = \ldots$

$x = -2: \quad 8 = -2C\left(-2 - \frac{2}{3}\right) \longrightarrow C = \ldots$

$x = 0: \quad 4 = A\left(-\frac{2}{3}\right) \cdot 2 \longrightarrow A = \ldots$

Bem. : <u>Ansatz für doppelte NS</u>:

Bsp: $\displaystyle\int \frac{3z^2+1}{(z+1)(z-5)^2}\,dz$

PBZ: $\displaystyle\frac{3z^2+1}{(z+1)(z-5)^2} = \frac{A}{z+1} + \frac{B}{z-5} + \frac{C}{(z-5)^2}$