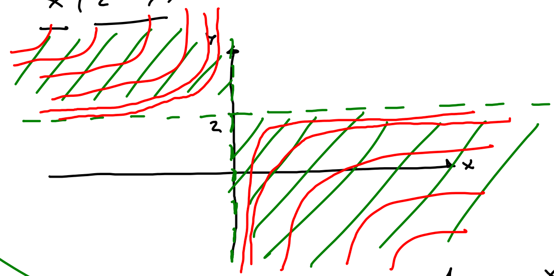
Nachbesprechung Prüfung 2

① (i) $f(x,y) = \log(x(2-y))$

$\longrightarrow$ benötigen $x(2-y) > 0 \longrightarrow$ entweder $x > 0$ & $2-y > 0 \rightarrow 2 > y$

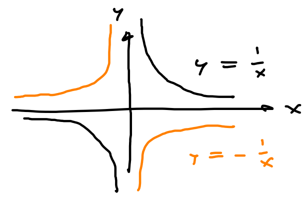


oder $x < 0$ & $2-y < 0 \rightarrow 2 < y$

(ii) $\log(x(2-y)) = C \quad / \, e^{(\dots)}$

$x(2-y) = \tilde{C} \; (>0) \longrightarrow 2-y = \frac{\tilde{C}}{x}$

$\longrightarrow \boxed{y = 2 - \frac{\tilde{C}}{x}}$

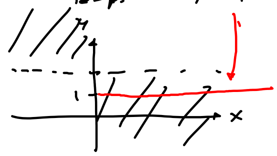


(iii) Werte des $\log(..)$: $\mathbb{R}$

$\longrightarrow$ Werte von $f(x,y) = \log(x(2-y))$: $\mathbb{R}$

Bsp.: $y = 1 \longrightarrow f(x,1) = \log(x)$ mit $x \in (0,\infty)$

$\longrightarrow f(x,1) \in \mathbb{R}$



② $f(x,y) = x^2 - xy + y^2 - y$

(i) $D_{\vec{e}} f(1,-1)$ am grössten : in Richtung von $\vec{\nabla} f(1,-1)$

$$\vec{\nabla} f = \begin{pmatrix} 2x - y \\ -x + 2y - 1 \end{pmatrix} \longrightarrow \vec{\nabla} f(1,-1) = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$\longrightarrow \underline{\vec{e}_{(a)} = \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}}$$

$D_{\vec{e}} f(1,-1) = \vec{e} \cdot \vec{\nabla} f(1,-1)$

$$= \begin{pmatrix} e_x \\ e_y \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \underline{3e_x - 4e_y \stackrel{!}{=} 4} \longrightarrow e_y = \frac{3}{4} e_x - 1$$

$$\& \quad \underline{e_x^2 + e_y^2 \stackrel{!}{=} 1}$$

$$e_x^2 + \left(\frac{3}{4} e_x - 1\right)^2 = 1$$

$$e_x^2 + \frac{9}{16} e_x^2 - \frac{3}{2} e_x + \cancel{1} = \cancel{1}$$

$$\frac{25}{16} e_x^2 = \frac{3}{2} e_x \longrightarrow \underline{e_x = 0} \; ; \; \underline{e_y = -1}$$

oder : $\frac{25}{16} e_x = \frac{3}{2} \rightarrow \underline{e_x = \frac{24}{25}}$; $e_y = \frac{3}{4} \cdot \frac{24}{25} - 1 = \frac{18}{25} - 1 = \underline{-\frac{7}{25}}$

I.e. $\underline{\vec{e}_{(b)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}}$ oder $\underline{\vec{e}_{(b)} = \begin{pmatrix} \frac{24}{25} \\ -\frac{7}{25} \end{pmatrix}}$

(ii) $\nabla f = \begin{pmatrix} 2x-y \\ -x+2y-1 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \begin{cases} 2x = y \longrightarrow 2y = 4x \\ -x+2y = 1 \end{cases}$

$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$

$-x + 4x = 1 \longrightarrow 3x = 1 \rightarrow \underline{x = \frac{1}{3}}$

$\longrightarrow \underline{y = \frac{2}{3}}$

I.e. $\underline{(x_0, y_0) = \left(\frac{1}{3}, \frac{2}{3}\right)}$

$f'' = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \longrightarrow \det(f'' - \lambda \mathbb{1}) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix}$

$f'' = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$

$= (2-\lambda)^2 - 1$

$= 4 - 4\lambda + \lambda^2 - 1$

$= \underbrace{\lambda^2 - 4\lambda + 3}_{(\lambda-1)(\lambda-3)} \overset{!}{=} 0$

$\longrightarrow \begin{matrix} \lambda_+ = 1 \\ \lambda_- = 3 \end{matrix} \longrightarrow$ \underline{lok. Min.}

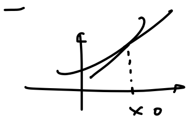
(iii) $\nabla f = \begin{pmatrix} 2x-y \\ -x+2y-1 \end{pmatrix} \longrightarrow \vec{n}_1 = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ -1 \end{pmatrix} = \begin{pmatrix} 2x-y \\ -x+2y-1 \\ -1 \end{pmatrix}$: Normalenvekt. der Tang.-Ebene an $z = f(x,y)$

Ebene: $y + z = 17 \longrightarrow \vec{n}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$: Norm.-Vekt. an Ebene $y + z = 17$

(Allg.: $ax + by + cz = d$, $\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$)

$\longrightarrow$ Bed.: $\vec{n}_1 = k\,\vec{n}_2$

i.e. $\begin{cases} 2x - y = 0 & (1) \\ -x + 2y - 1 = k & (2) \\ -1 = k \longrightarrow \underline{k = -1} \end{cases}$



$(1) \longrightarrow 2x = y$

in (2): $-x + 4x - 1 = -1$

i.e. $3x = 0 \rightarrow \underline{x = 0} \rightarrow \underline{y = 0}$

I.e. $\underline{(x_1, y_1) = (0, 0)}$

$y = y_0 + \frac{df}{dx}(x_0)(x - x_0)$

oder: $y - y_0 = \frac{df}{dx}(x_0)(x - x_0)$

in 2D: $z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$

$\updownarrow$

$ax + by + cz = d \longrightarrow \vec{n} = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ -1 \end{pmatrix}$

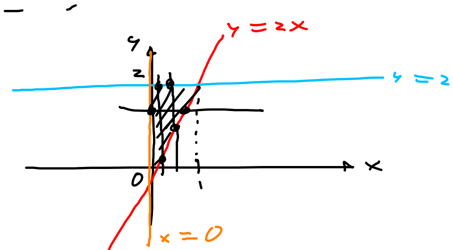
③ (i) $z = 6 - 5x^2 \longrightarrow$ Graph von $f(x,y) = 6 - 5x^2$

$y = 2x$

$y = 2$

$x = 0$

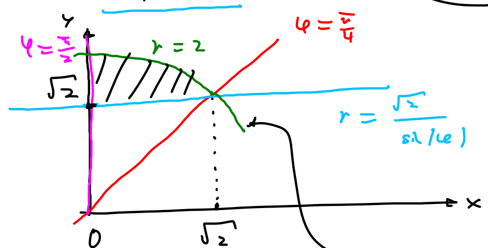
$z = 0 \longrightarrow$ x-y-Ebene



$$\longrightarrow \quad V = \int_{x=0}^{x=1} \int_{y=2x}^{y=2} (6 - 5x^2)\,dy\,dx$$

oder: $$V = \int_{y=0}^{y=2} \int_{x=0}^{y/2} (6 - 5x^2)\,dx\,dy$$

(ii) $$\int_{\varphi=\pi/4}^{\varphi=\pi/2} \int_{r=\sqrt{2}/\sin(\varphi)}^{r=2} r^3\,dr\,d\varphi = \int_{x=0}^{x=\sqrt{2}} \int_{y=\sqrt{2}}^{y=\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx$$



$r = \frac{\sqrt{2}}{\sin(\varphi)}$

i.e. $\underbrace{r\sin(\varphi)}_{=y} = \sqrt{2}$

i.e. $y = \sqrt{2}$

$x^2+y^2 = 4 \longrightarrow y = \sqrt{4-x^2}$

$r^2 = x^2+y^2$

$\iint \dots r\,dr\,d\varphi = \iint \dots dy\,dx$

[Bsp.: $$\iint\limits_{\dots}^{\dots} dr\,d\varphi = \iint\limits_{\dots}^{\dots} \frac{r\,dr\,d\varphi}{r} = \iint \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx$$]

(4) $\sin(0.01)\cos(0.99\pi) \approx ?$

Idee: $f(x,y) = \sin(x)\cos(y)$

$\underline{f(0,\pi)} = \sin(0)\cos(\pi) = 0\cdot(-1) = \underline{0}$

Benutze: $f(x,y) \approx \underbrace{f(x_0,y_0)}_{=0} + \left(\frac{\partial f}{\partial x}\right)_0 (x-x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y-y_0)$

$\frac{\partial f}{\partial x} = \cos(x)\cos(y) \longrightarrow \left(\frac{\partial f}{\partial x}\right)_0 = \cos(0)\cos(\pi) = 1\cdot(-1) = -1$

$\frac{\partial f}{\partial y} = -\sin(x)\sin(y) \longrightarrow \left(\frac{\partial f}{\partial y}\right)_0 = -\sin(0)\sin(\pi) = 0$

$\longrightarrow \underline{\sin(0.01)\cos(0.99\pi)} = f(0.01,\, 0.99\pi)$

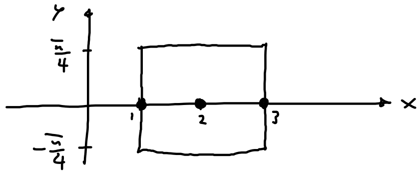
$\approx f(0,\pi) + \frac{\partial f}{\partial x}(0,\pi)\cdot 0.01 + \frac{\partial f}{\partial y}(0,\pi)\cdot(-0.01\pi)$

$= 0 + (-1)\cdot 0.01 + 0\cdot(-0.01\pi)$

$= \underline{-0.01}$

(5) $f(x,y) = (4x - x^2)\cos(y)$

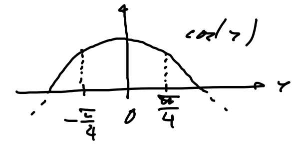
$G = \{(x,y)\in\mathbb{R}^2 :\ 1 \le x \le 3;\ -\frac{\pi}{4} \le y \le \frac{\pi}{4}\}$



Innere Pkte.:

$$\vec{\nabla} f = \begin{pmatrix} (4-2x)\cos(y) \\ -(4x-x^2)\sin(y) \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

$(1) \longrightarrow (4-2x)\underbrace{\cos(y)}_{\neq 0} = 0$



$\longrightarrow 4-2x=0$

$\longrightarrow \underline{x=2}$

in (2) $\longrightarrow -(8-4)\sin(y)=0$

i.e. $\sin(y)=0 \longrightarrow \underline{y=0}$

$\longrightarrow \underline{(x_0,y_0)=(2,0)} \longrightarrow \underline{f(2,0)=4}$

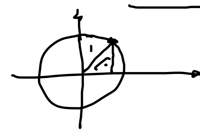
Rand:

Links: $g(y)=f(1,y)=3\cos(y) \longrightarrow$ max. bei $y=0 \longrightarrow \underline{f(1,0)=3}$

rechts: $g(y)=f(3,y)=3\cos(y)$ $\quad\dots \longrightarrow \underline{f(3,0)=3}$

oben: $g(x)=f(x,\frac{\pi}{4})=(4x-x^2)\cos(\frac{\pi}{4})$

$= (4x-x^2)\frac{1}{\sqrt{2}}$



max. bei $\frac{dg}{dx}(x)=(4-2x)\frac{1}{\sqrt{2}} \overset{!}{=} 0 \longrightarrow \underline{x=2} \longrightarrow \underline{f(2,\frac{\pi}{4})=\frac{4}{\sqrt{2}}}$

unten: $g(x)=f(x,-\frac{\pi}{4})=(4x-x^2)\frac{1}{\sqrt{2}} \longrightarrow \dots \longrightarrow x=2 \longrightarrow \underline{f(2,-\frac{\pi}{4})=\frac{4}{\sqrt{2}}}$

Eckpkte: $\underline{f(1,\frac{\pi}{4})=\frac{3}{\sqrt{2}}}$

$\underline{f(1,-\frac{\pi}{4})=\frac{3}{\sqrt{2}}}$

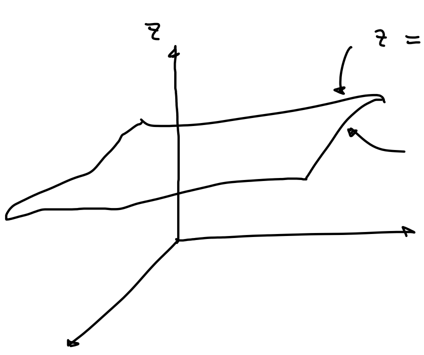
$\underline{f(3,\frac{\pi}{4})=\frac{3}{\sqrt{2}}}$

$\underline{f(3,-\frac{\pi}{4})=\frac{3}{\sqrt{2}}}$

$\longrightarrow$ max. bei $(2,0)$: max $=4$

min. in Ecken, min $=\frac{3}{\sqrt{2}}$

(b) (i)



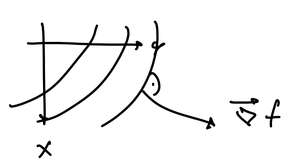
$z=f(x,y)$

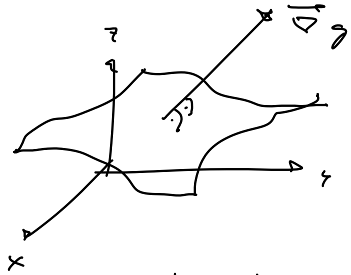
Graph besteht aus Pkten. $(x,y,f(x,y))$

$\underline{g(x,y,z)=f(x,y)-z}$

$\longrightarrow g(x,y,z)=0$ ist $z=f(x,y)$

(ii) $(x_0,y_0,f(x_0,y_0))$





$$\vec{\nabla} g = \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ -1 \end{pmatrix} \longrightarrow \vec{\nabla} g(x_0,y_0,f(x_0,y_0)) = \begin{pmatrix} \left(\frac{\partial f}{\partial x}\right)_0 \\ \left(\frac{\partial f}{\partial y}\right)_0 \\ -1 \end{pmatrix} = \vec{n}$$

Ebene: $\vec{n} \cdot \begin{pmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{pmatrix} = 0$

$$\rightarrow \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0) = z - z_0$$