Flächenmomente

Biegemoment im Querschnitt

Einführung (siehe auch Strukturmechanik)

Gleichung für Biegelinie Balkens $y(x)$: $\frac{d^2y}{dx^2}(x) = \frac{1}{EI} M(x)$

Krümmung

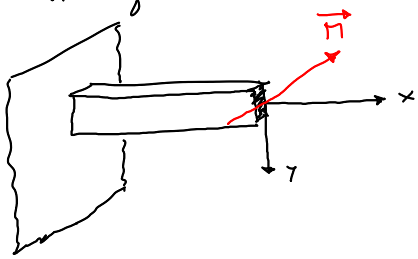
E-Modul (Material)

Geometrie: Flächenmoment

Welche Dimension (Einheiten) hat I:

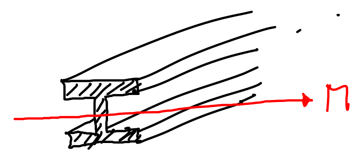
$\left[\frac{d^2y}{dx^2}\right] = \frac{1}{m}$; $[E] = \frac{N}{m^2}$; $[M] = Nm \rightarrow \underline{[I]} = \left[\frac{M}{E\frac{d^2y}{dx^2}}\right] = \frac{Nm\, m^2\, m}{N} = \underline{m^4}$

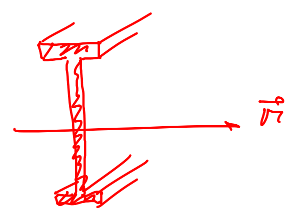
I ist geometrischer Widerstand des Querschnitts gegen Biegung



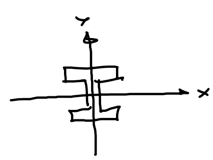
Grosser Widerstand → viel Fläche weg von der Achse entlang welcher das Moment verläuft.

→ Typischer Biegeträger:





Wir verwenden x-y-Koord. im Querschnitt:



Def:

$$I_x = \iint_\Omega y^2\, dA$$

$$I_y = \iint_\Omega x^2\, dA$$

Axiale Flächenmomente

$$I_{xy} = -\iint_\Omega xy\, dA$$

Deviationsmoment

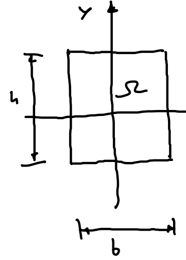
⚠ Flächenmomente beziehen sich auf Koord.-Achsen.

Bem.: (i) Konvention Strukturmech.

- y-z-Achsen im Querschnitt
- $\iint_\Omega xy\, dA$ an Stelle von $-\iint_\Omega xy\, dA$

(ii) Formeln <u>nicht</u> auswendig!

Bsp:



Achsen durch SP

$$\underline{I_x} = \iint_\Omega y^2 \, dA$$

$$= \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 \, dy \, dx = \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{y^3}{3} \Big|_{-\frac{h}{2}}^{\frac{h}{2}} dx = \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{h^3}{12} \, dx$$

$$= \frac{h^3}{12} \int_{-\frac{b}{2}}^{\frac{b}{2}} dx = \frac{h^3}{12} \, x \Big|_{-\frac{b}{2}}^{\frac{b}{2}} = \underline{\frac{h^3 b}{12}} \; ; \; \underline{I_y = \frac{h b^3}{12}}$$

$$\underline{I_{xy}} = -\int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} xy \, dy \, dx = -\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{xy^2}{2} \Big|_{-\frac{h}{2}}^{\frac{h}{2}} dx = \int_{-\frac{b}{2}}^{\frac{b}{2}} 0 \, dx = \underline{0}$$

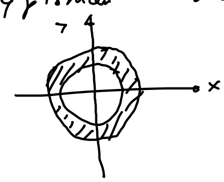
Allgemein gilt: $I_{xy} = 0$ wenn x- & y-Achse Symmetrieachsen!

Polares Flächenmoment

Def. $I_p = \iint_\Omega (x^2 + y^2) \, dA$

Bei kreisförmigem Querschnitt: Widerstand gegen Torsion.

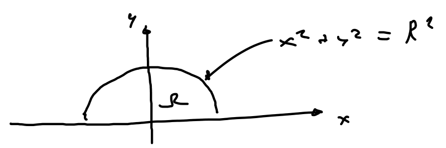
Typischer Torsionsquerschnitt:



Bem: (i) Ist bezogen auf Ursprung

(ii) $I_p = I_x + I_y$

Bsp:



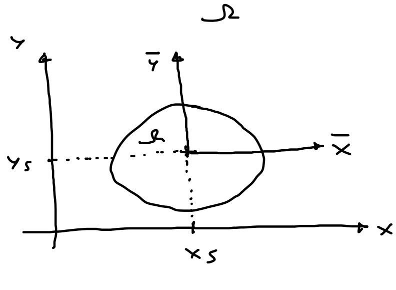
$$\underline{I_p} = \iint_\Omega (x^2 + y^2) \, dA = \int_{\varphi=0}^{\pi} \int_{r=0}^{R} r^2 \, r \, dr \, d\varphi$$

$$= \int_0^{\pi} \frac{r^4}{4} \Big|_0^R d\varphi = \underline{\frac{\pi R^4}{4}}$$

Satz von Steiner:

Sei $I_x = \iint_\Omega y^2 \, dA$ Flächenmoment bzgl. Achsen durch Ursprung

$I_{xs} = \iint_\Omega \bar{y}^2 \, dA$ —"— SP



Es gilt: $I_{xs} = I_x - |\Omega| \, y_s^2$

Beweis:

$$\bar{x} = x - x_s \qquad \bar{y} = y - y_s \longrightarrow I_{xs} = \iint_\Omega \bar{y}^2 \, dA$$

$$= \iint_\Omega (y - y_s)^2 \, dA$$

$$= \iint_\Omega (y^2 - 2 y y_s + y_s^2) \, dA$$

$$= \iint_\Omega y^2 dA - \iint_\Omega 2 y y_s \, dA + \iint_\Omega y_s^2 \, dA$$

$$= I_x - 2 y_s \underbrace{\iint_\Omega y \, dA}_{= y_s |\Omega|} + y_s^2 \underbrace{\iint_\Omega dA}_{= |\Omega|}$$

$$= I_x - 2 y_s^2 |\Omega| + y_s^2 |\Omega|$$

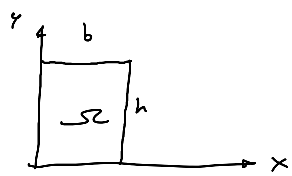
$$= \underline{I_x - y_s^2 |\Omega|} \qquad \square$$

Analog gilt:

$$I_{ys} = I_y - x_s^2 |\Omega|$$

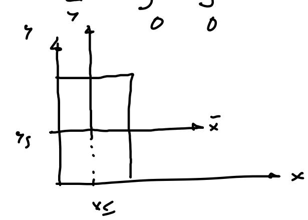
$$I_{xys} = I_{xy} + x_s y_s |\Omega|$$

Bsp:



$$\underline{I_x} = \iint_\Omega y^2 \, dA$$

$$= \int_0^b \int_0^h y^2 \, dy \, dx = \int_0^b \frac{h^3}{3} dx = \underline{\frac{b h^3}{3}}$$



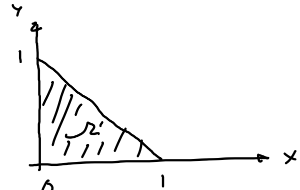
$$x_s = \frac{b}{2} \; ; \; y_s = \frac{h}{2}$$

$$\longrightarrow \underline{I_{xs}} = I_x - y_s^2 |\Omega|$$

$$= \frac{b h^3}{3} - \frac{h^2}{4} b h = \underline{\frac{b h^3}{12}}$$

Aufgabe:

Gegeben ist Querschnitt:



Man bestimme:

(i) I_x, I_y, I_{xy}

(ii) $|\Omega|, (x_s, y_s)$

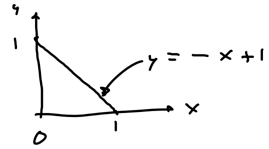
(iii) I_{xs}, I_{ys}, I_{xys}

(iv) Hauptflächenmomente $\longrightarrow$ EW von I

(v) Hauptachsen $\longrightarrow$ EV von I

Idee:
$$I = \begin{pmatrix} I_{xs} & I_{xys} \\ I_{xys} & I_{ys} \end{pmatrix}$$

Lsg.: (i)



$$\underline{I_x} = \int_0^1 \int_0^{-x+1} y^2 \, dy \, dx$$

$$= \int_0^1 \frac{y^3}{3} \Big|_0^{-x+1} dx = \int_0^1 \frac{(-x+1)^3}{3} dx$$

$$= -\frac{(-x+1)^4}{12} \Big|_0^1 = \underline{\frac{1}{12}}$$

$$\underline{I_y} \underset{\text{Symmetrie}}{=} \underline{\frac{1}{12}}$$

$$\underline{I_{xy}} = -\int_0^1 \int_0^{-x+1} xy \, dy \, dx = -\int_0^1 \frac{xy^2}{2} \Big|_0^{-x+1} dx$$

$$= -\int_0^1 x \frac{(-x+1)^2}{2} dx$$

$$= -\frac{1}{2} \int_0^1 (x^3 - 2x^2 + x) \, dx$$

$$= -\frac{1}{2} \left(\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} \right) \Big|_0^1$$

$$= -\frac{1}{2} \left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2} \right) = \underline{-\frac{1}{24}}$$

(ii) $\underline{|\Omega| = \frac{1}{2}}$ $\left(\text{oder } \underline{|\Omega|} = \int_0^1 \int_0^{-x+1} dy \, dx = \dots = \underline{\frac{1}{2}} \right)$

$$\underline{x_s} = \frac{1}{|\Omega|} \int_0^1 \int_0^{-x+1} x \, dy \, dx = \frac{1}{|\Omega|} \int_0^1 x(-x+1) \, dx$$

$$= \frac{1}{|\Omega|} \left(-\frac{x^3}{3} + \frac{x^2}{2} \right) \Big|_0^1$$

$$= \frac{1}{|\Omega|} \left(-\frac{1}{3} + \frac{1}{2} \right) = 2 \cdot \frac{1}{6} = \underline{\frac{1}{3}}$$

Symm.

$$\underline{y_s = \frac{1}{3}}$$

(iii) $\underline{I_{x_s}} = I_x - y_s^2 |\Omega| = \frac{1}{12} - \frac{1}{9} \cdot \frac{1}{2} = \underline{\frac{1}{36}}$

$\underline{I_{y_s}} \underset{\text{Sym.}}{=} \underline{\frac{1}{36}}$; $\underline{I_{x_s y_s}} = I_{xy} + x_s y_s |\Omega| = \dots = \underline{\frac{1}{72}}$

(iv) $I_s = \begin{pmatrix} \frac{1}{36} & \frac{1}{72} \\ \frac{1}{72} & \frac{1}{36} \end{pmatrix} \longrightarrow$ EW: $\left. \begin{array}{l} \lambda_+ = \frac{3}{72} \\ \lambda_- = \frac{1}{72} \end{array} \right\}$ Hauptflächenmomente

(v) $ER(\lambda_+) = t \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $t \in \mathbb{R}$; $ER(\lambda_-) = t \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \longrightarrow$