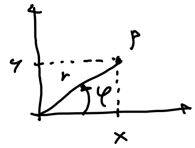
Doppelintegrale Polarkoord.

Polarkoord.

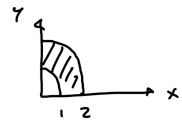


$$\begin{cases} x = r\cos(\varphi) \\ y = r\sin(\varphi) \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \tan(\varphi) = \frac{y}{x} \end{cases}$$

Gebiete in x-y-Ebene:

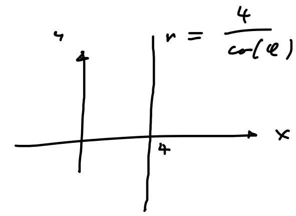
$1 \leq r \leq 2 \; ; \; 0 \leq \varphi \leq \frac{\pi}{2}$



Funktionen:

(i) $r = \dfrac{4}{\cos(\varphi)}$; $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\longrightarrow \underbrace{r\cos(\varphi)}_{=x} = 4$ i.e. $x = 4$



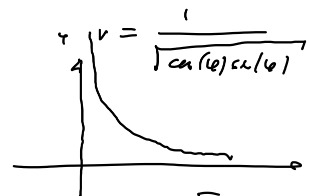
(ii) $r = \dfrac{1}{\sqrt{\cos(\varphi)\sin(\varphi)}}$; $\varphi \in (0, \frac{\pi}{2})$

$\longrightarrow r^2 = \dfrac{1}{\cos(\varphi)\sin(\varphi)}$

$\underbrace{r\cos(\varphi)}_{x}\,\underbrace{r\sin(\varphi)}_{y} = 1$

i.e. $xy = 1$

i.e. $y = \frac{1}{x}$



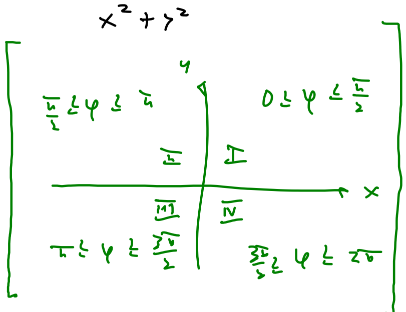
$\left[(x-x_0)^2 + (y-y_0)^2 = r^2 \right]$

(iii) $r = 6\sin(\varphi)$ $\longrightarrow$ $r^2 = 6\underbrace{r\sin(\varphi)}_{y}$ $\longrightarrow$ $x^2+y^2 = 6y$

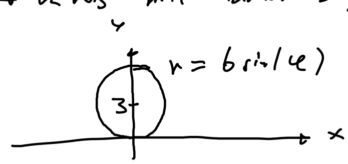
$\varphi \in [0, \pi]$ $\qquad$ $r^2 = x^2+y^2$

$\longrightarrow x^2+y^2-6y+9 = 0+9$

$x^2 + (y-3)^2 = 9$



$\longrightarrow$ Kreis mit Radius 3, Zentrum bei $(0,3)$



(iv) Aufgabe: $r = 4\cos(\varphi)$

$\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

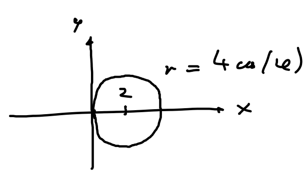
Lösg.: $r^2 = 4\underbrace{r\cos(\varphi)}_{=x}$

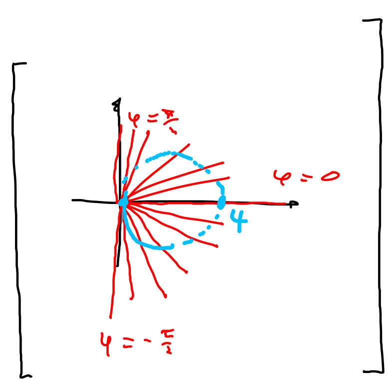
$r^2 = x^2+y^2$

i.e. $x^2 - 4x + y^2 = 0$

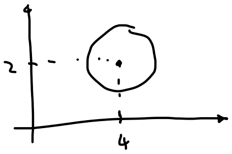
$x^2 - 4x + 4 + y^2 = 4$

$(x-2)^2 + y^2 = 4$





Bem:



$$(x-4)^2 + (y-2)^2 = 1$$

$$x^2 - 8x + 16 + y^2 - 4y + 4 = 1$$

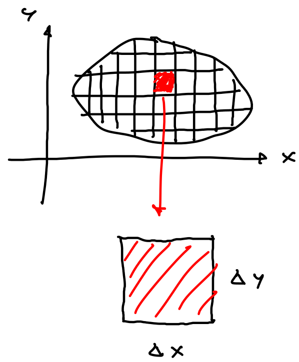
$$x^2 + y^2 + 19 = 8x + 4y$$

$$\underbrace{r^2}_{x^2+y^2} + 19 = \underbrace{8 \underline{r\cos(\varphi)} + 4 \underline{r\sin(\varphi)}}_{x \qquad\qquad y}$$

Integrale

Flächenelement

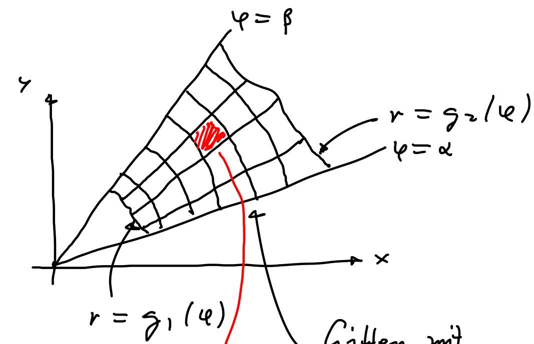
Kartesisch:



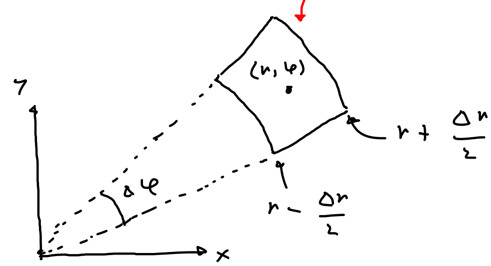
$$\longrightarrow \Delta A = \Delta x \Delta y$$

$$\longrightarrow \boxed{dA = dx\,dy}$$

polar:



Gitter mit φ = konst. & r = konst. Linien



$\Delta A =$

$$\left(r + \frac{\Delta r}{2}\right)^2 \pi \frac{\Delta \varphi}{2\pi} = \frac{\Delta \varphi}{2}\left(r + \frac{\Delta r}{2}\right)^2$$

$$\longrightarrow \underline{\Delta A} = \frac{\Delta\varphi}{2}\left(r + \frac{\Delta r}{2}\right)^2 - \frac{\Delta\varphi}{2}\left(r - \frac{\Delta r}{2}\right)^2$$

$$= \frac{\Delta\varphi}{2}\left(\left(r + \frac{\Delta r}{2}\right)^2 - \left(r - \frac{\Delta r}{2}\right)^2\right) = \frac{\Delta\varphi}{2} 2r\Delta r = \underline{r \,\Delta r\, \Delta\varphi}$$

$$\longrightarrow \boxed{dA = r\,dr\,d\varphi}$$

Doppelintegral in Polarkoord.

$$\boxed{\iint_\Omega f(r,\varphi)\,dA = \int_{\varphi=\alpha}^{\varphi=\beta} \int_{r=g_1(\varphi)}^{r=g_2(\varphi)} f(r,\varphi)\,\boxed{r}\,dr\,d\varphi}$$

Traf: kartesisch $\longrightarrow$ polar

$$\iint_\Omega f(x,y)\,dA = \int_\alpha^\beta \int_{g_1(\varphi)}^{g_2(\varphi)} f(r\cos(\varphi), r\sin(\varphi))\, r\,dr\,d\varphi$$

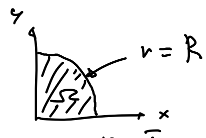
Fläche:

$$\iint_\Omega dA = \int_\alpha^\beta \int_{g_1(\varphi)}^{g_2(\varphi)} r\,dr\,d\varphi$$

Schwerpkt:

$$x_S = \frac{1}{|\Omega|}\iint_\Omega x\,dA = \frac{1}{|\Omega|}\int_\alpha^\beta \int_{g_1(\varphi)}^{g_2(\varphi)} r^2\cos(\varphi)\,dr\,d\varphi$$

$$y_S = \frac{1}{|\Omega|}\iint_\Omega y\,dA = \frac{1}{|\Omega|}\iint_{\dots}^{\dots} r^2\sin(\varphi)\,dr\,d\varphi$$

Bsp: SP von  Fläche: $|\Omega| = \frac{R^2\pi}{4}$

$$\iint_\Omega x\,dA = \int_{\varphi=0}^{\varphi=\frac{\pi}{2}} \int_{r=0}^{r=R} r^2\cos(\varphi)\,dr\,d\varphi$$

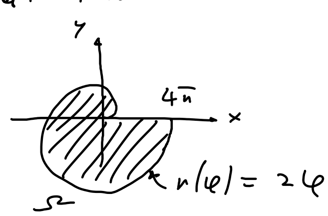
$$= \int_0^{\pi/2} \frac{r^3}{3}\Big|_0^R \cos(\varphi)\,d\varphi = \int_0^{\pi/2} \frac{R^3}{3}\cos(\varphi)\,d\varphi$$

$$= \frac{R^3}{3}\sin(\varphi)\Big|_0^{\pi/2} = \frac{R^3}{3}$$

$$\longrightarrow x_S = \frac{1}{|\Omega|}\iint_\Omega x\,dA = \frac{4}{R^2\pi}\cdot\frac{R^3}{3} = \underline{\frac{4R}{3\pi}}$$

$\underline{y_S = x_S}$ (Symmetrie!)

Aufgabe: Ges: Fläche innerhalb von $r(\varphi) = 2\varphi$; $\varphi \in [0, 2\pi]$.

Lsg.: 

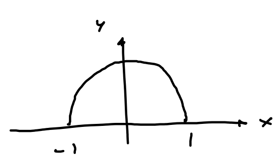
$$\underline{|\Omega|} = \iint_\Omega dA = \int_0^{2\pi}\int_0^{2\varphi} r\,dr\,d\varphi$$

$$= \int_0^{2\pi} \frac{r^2}{2}\Big|_0^{2\varphi} d\varphi$$

$$= \int_0^{2\pi} 2\varphi^2\,d\varphi = \frac{2}{3}\varphi^3\Big|_0^{2\pi}$$

$$= \underline{\frac{16\pi^3}{3}}$$

Aufgabe: $\iint_\Omega e^{x^2+y^2}\,dx\,dy$ mit $\Omega = \{(x,y)\in\mathbb{R}^2 : -1 \le x \le 1 ;\ 0 \le y \le \sqrt{1-x^2}\}$

Lsg: 

$$\iint_\Omega e^{x^2+y^2}\,dx\,dy = \int_0^\pi\int_0^1 e^{r^2} r\,dr\,d\varphi = \int_0^\pi \frac{1}{2}e^{r^2}\Big|_0^1 d\varphi$$

$$= \int_0^\pi \frac{1}{2}(e-1)\,d\varphi = \underline{\frac{\pi}{2}(e-1)}$$