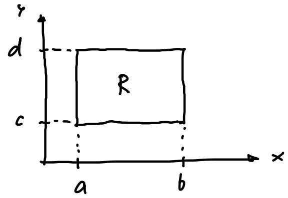
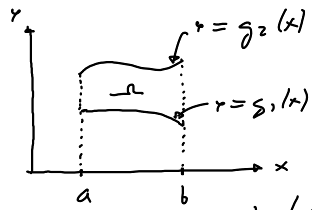
Mehrfachintegrale

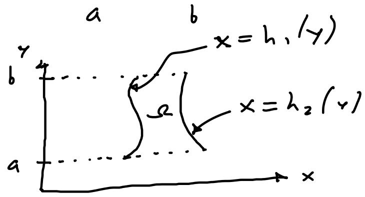
Wiederholung:



$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$



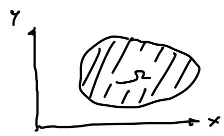
$$\iint_\Omega f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$



$$\iint_\Omega f(x,y)\,dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

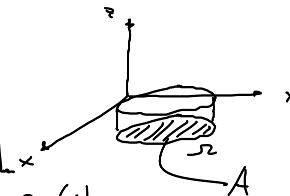
Erste Anwendungen:

Flächeninhalt



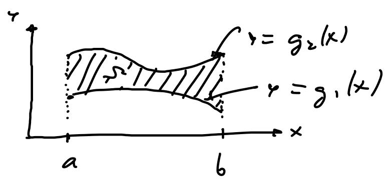
$$\boxed{A = \iint_\Omega dA}$$

[Interpretation:



]

Falls Gebiet gegeben durch:



$$\boxed{A} = \iint_\Omega dA = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx$$

$$= \int_a^b y \Big|_{g_1(x)}^{g_2(x)} dx$$

$$\boxed{= \int_a^b \big(g_2(x) - g_1(x)\big)\,dx}$$

[eine uns bekannte Formel!]

Notation: Für $\Omega \subset \mathbb{R}^2$: $|\Omega| = \iint_\Omega dA$

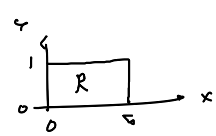
Durchschnitt: von $f(x,y)$ in Ω:

$$\boxed{\bar{f} = \frac{1}{|\Omega|} \iint_\Omega f(x,y)\,dA}$$

[Erinnerung: 1D:

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\,dx$$
]

Aufgabe: Man bestimme Durchschnitt von $f(x,y) = x\cos(xy)$ in $R = \{(x,y) \in \mathbb{R}^2 :\ 0 \leq x \leq \pi;\ 0 \leq y \leq 1\}$.



Lösg.: $|\Omega| = \pi$

$$\iint_\Omega f(x,y)\,dA = \int_0^\pi \int_0^1 x\cos(xy)\,dy\,dx = \int_0^\pi \sin(xy)\Big|_0^1 dx = \int_0^\pi \sin(x)\,dx = -\cos(x)\Big|_0^\pi$$

$$= 1 - (-1) = 2$$

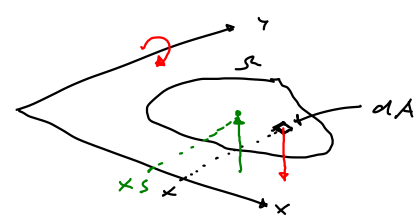
$$\longrightarrow \bar{f} = \frac{1}{|\Omega|} \iint_\Omega f(x,y)\,dA = \frac{2}{\pi}$$

Schwerpkt. einer Fläche:

Schwerpktskoord. (x_S, y_S) eines Gebiets Ω:

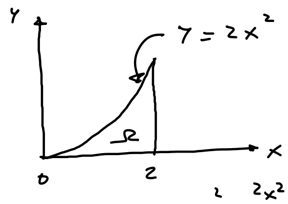
$$\boxed{\begin{aligned} x_S &= \frac{1}{|\Omega|}\iint_\Omega x\,dA \\ y_S &= \frac{1}{|\Omega|}\iint_\Omega y\,dA \end{aligned}}$$

Interpretation:



$$x_S|\Omega| = \iint_\Omega x\,dA$$

Aufgabe: Ges: Schwerpkt. von:



Lösg.:

$$\underline{|\Omega|} = \iint_\Omega dA = \int_0^2\int_0^{2x^2} dy\,dx = \int_0^2 2x^2\,dx = \frac{2x^3}{3}\Big|_0^2 = \underline{\frac{16}{3}}$$

$$\underline{x_S} = \frac{1}{|\Omega|}\iint_\Omega x\,dA = \frac{3}{16}\int_0^2\int_0^{2x^2} x\,dy\,dx = \frac{3}{16}\int_0^2 xy\Big|_0^{2x^2}dx = \frac{3}{16}\int_0^2 2x^3\,dx = \frac{3}{16}\cdot\frac{x^4}{2}\Big|_0^2$$

$$= \underline{\frac{3}{2}}$$

$$\underline{y_S} = \frac{1}{|\Omega|}\iint_\Omega y\,dA = \frac{3}{16}\int_0^2\int_0^{2x^2} y\,dy\,dx = \frac{3}{16}\int_0^2 \frac{y^2}{2}\Big|_0^{2x^2}dx = \frac{3}{16}\int_0^2 2x^4\,dx$$

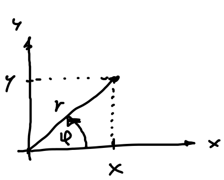
$$= \frac{3}{16}\cdot\frac{2x^5}{5}\Big|_0^2$$

$$= \frac{3\cdot 2^6}{2^4\cdot 5} = \frac{3\cdot 2^2}{5} = \underline{\frac{12}{5}}$$

$$\longrightarrow \underline{(x_S, y_S) = \left(\frac{3}{2}, \frac{12}{5}\right)}$$

Doppelintegrale in Polarkoord.

Erinnerung: Polarkoord.



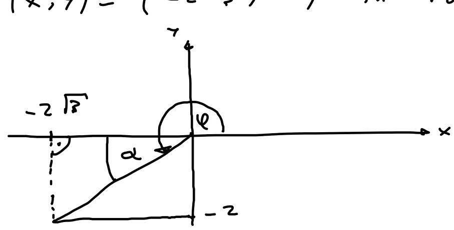
$$\begin{cases} x = r\cos(\varphi) \\ y = r\sin(\varphi) \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \tan(\varphi) = \frac{y}{x} \longrightarrow \varphi \end{cases}$$

Menschenverstand!

oder: $\varphi = \operatorname{atan2}\left(\frac{y}{x}\right)$

Bsp: $(x,y) = (-2\sqrt{3}, -2)$ in Polarkoord.:



$$\underline{r} = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{12+4} = \underline{4}$$

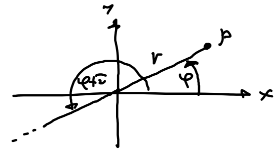
$$\tan(\alpha) = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \longrightarrow \alpha = \frac{\pi}{6}$$

$$\longrightarrow \underline{\varphi} = \alpha + \pi = \underline{\frac{7\pi}{6}}$$

I.e. $\underline{(r,\varphi) = \left(4, \frac{7\pi}{6}\right)}$

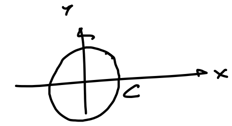
Bem.: (i) φ nicht eindeutig: (r,φ) ist gleicher Pkt. wie $(r, \varphi + 2\pi)$.

(ii) $r < 0$ erlaubt: (r,φ) ist gleicher Pkt. wie $(-r, \varphi + \pi)$

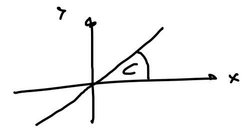


Polargleichungen / Graphen

Bsp: (i) $r = C \longrightarrow$ Graph: Kreis mit Radius C:

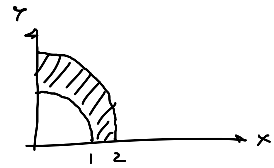


(ii) $\varphi = C \longrightarrow$ Graph: Gerade durch Ursprung:

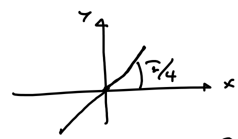


Gebiete:

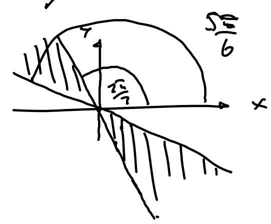
Bsp: (i) $1 \le r \le 2 \,;\; 0 \le \varphi \le \frac{\pi}{2}$



(ii) $-3 \le r \le 2 \,;\; \varphi = \frac{\pi}{4}$



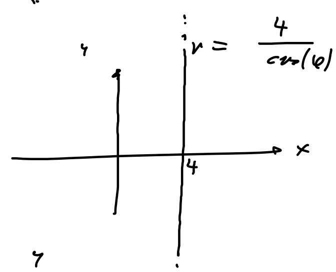
(iii) $\frac{2\pi}{3} \le \varphi \le \frac{5\pi}{6}$



Funktionen $r(\varphi)$:

(i) $r = \dfrac{4}{\cos(\varphi)} \longrightarrow \underbrace{r\cos(\varphi)}_{=x} = 4 \longrightarrow x = 4$

$\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



(ii) $r = -\dfrac{3}{\sin(\varphi)} \longrightarrow$

$\varphi \in (0, \pi)$