Mehrfachintegrale

Def.: $\iint\limits_R f(x,y)\,dA = \lim\limits_{\substack{n\to\infty \\ \|P_n\|\to 0}} \sum\limits_{k=1}^{n} f(x_k,y_k)\,\Delta A_k$

$R = \{(x,y)\in\mathbb{R}^2 : a \le x \le b;\ c \le y \le d\}$.

Berechnung:

$$\boxed{\iint\limits_R f(x,y)\,dA = \int\limits_{x=a}^{x=b}\int\limits_{y=c}^{y=d} f(x,y)\,dy\,dx = \int\limits_{y=c}^{y=d}\int\limits_{x=a}^{x=b} f(x,y)\,dx\,dy}$$

Aufgabe: $\iint\limits_R f(x,y)\,dA$ mit: $f(x,y) = 100 - 6x^2y$

$R = \{(x,y)\in\mathbb{R}^2 : 0 \le x \le 2;\ -1 \le y \le 1\}$.

Lösg.:

$$\int_0^2\int_{-1}^1 (100-6x^2y)\,dy\,dx = \int_0^2 \left(100y - 3x^2y^2\right)\Big|_{-1}^{1}\,dx$$

$$= \int_0^2 \Big((100-3x^2) - (-100-3x^2)\Big)\,dx$$

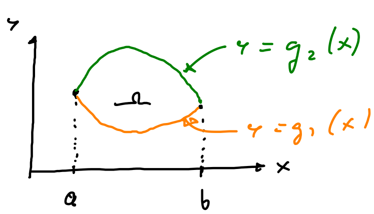
$$= \int_0^2 200\,dx = 200x\,\Big|_0^2 = \underline{400}$$

$$\int_{-1}^1\int_0^2 (100-6x^2y)\,dx\,dy = \int_{-1}^1 \left(100x - 2x^3y\right)\Big|_0^2\,dy$$

$$= \int_{-1}^1 (200-16y)\,dy = \left(200y - 8y^2\right)\Big|_{-1}^{1}$$

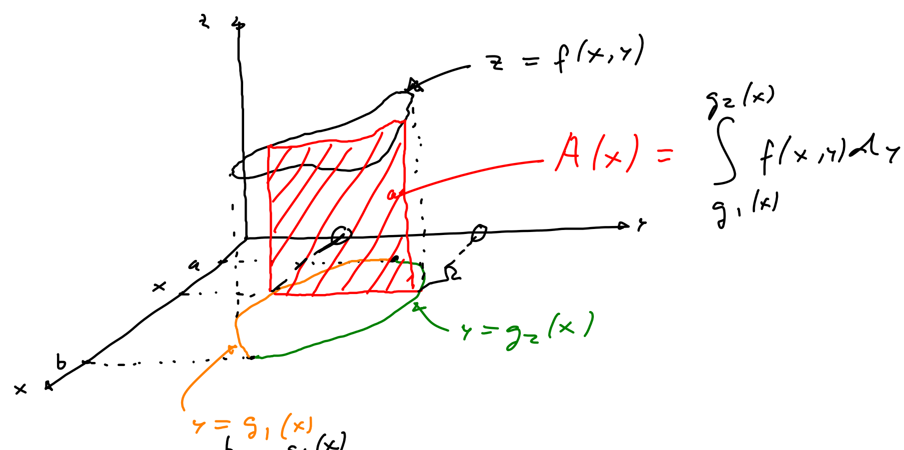
$$= (200-8) - (-200-8) = \underline{400}$$

Nicht-rechteckiger Integrationsbereich



Ges.: $\iint\limits_\Omega f(x,y)\,dA$

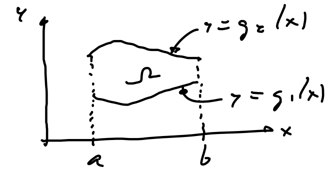
$g_1(x),\ g_2(x)$: geg. Fkt.



$$\longrightarrow \underline{V} = \int_a^b A(x)\,dx = \underline{\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx}$$

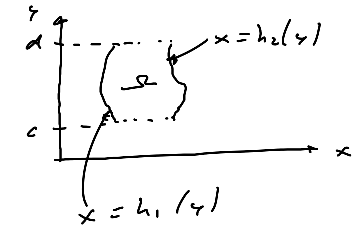
Falls Int.-Bereich Ω geg.: $\underline{\Omega = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b;\ g_1(x) \leq y \leq g_2(x)\}}$

$$\longrightarrow \quad \underline{\iint_\Omega f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx}$$

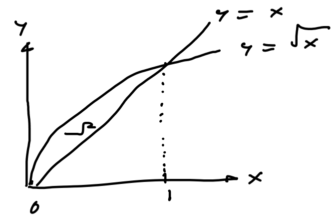


Falls Int.-Bereich Ω geg.: $\underline{\Omega = \{(x,y) \in \mathbb{R}^2 : c \leq y \leq d;\ h_1(y) \leq x \leq h_2(y)\}}$

$$\longrightarrow \quad \underline{\iint_\Omega f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy}$$



Bsp: $\quad I_{xy} = -\iint_\Omega xy\, dA \quad$ mit Ω:



Grenzen für x: $0 \leq x \leq 1$

Grenzen für y: $x \leq y \leq \sqrt{x}$

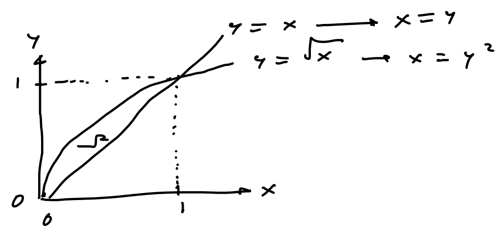
$$\longrightarrow \quad -\iint_\Omega yx\, dA = -\int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx = -\int_0^1 \frac{xy^2}{2}\Big|_x^{\sqrt{x}} dx$$

$$= -\int_0^1 \left(\frac{x^2}{2} - \frac{x^3}{2}\right) dx$$

$$= -\left(\frac{x^3}{6} - \frac{x^4}{8}\right)\Big|_0^1$$

$$= -\left(\frac{1}{6} - \frac{1}{8}\right) = \underline{-\frac{1}{24}}$$

Aufgabe: Wie obiges Bsp. jedoch mit vertauschten Int.-Reihenfolge.



Lösg:

$$\iint_\Omega f(x,y)\, dA = \int_{y=0}^{y=1} \int_{x=y^2}^{x=y} f(x,y)\, dx\, dy$$

$$= -\int_0^1 \int_{y^2}^{y} xy\, dx\, dy = -\int_0^1 \frac{x^2 y}{2}\Big|_{y^2}^{y} dy$$

$$= -\int_0^1 \left(\frac{y^3}{2} - \frac{y^5}{2}\right) dy$$

$$= -\left(\frac{y^4}{8} - \frac{y^6}{12}\right)\Big|_0^1$$

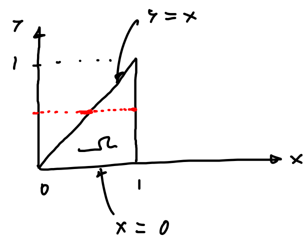
$$= -\left(\frac{1}{8} - \frac{1}{12}\right) = \underline{-\frac{1}{24}}$$

Aufgabe: Prisma: – Grundfläche in x-y-Ebene, begrenzt durch x-Achse & Geraden $y = x$, $x = 1$.

– Deckfläche: $z = 3 - x - y$

Ges.: Volumen

Lösg.:



$$V = \iint_\Omega (3 - x - y)\,dA$$

$$= \int_0^1 \int_0^x (3 - x - y)\,dy\,dx$$

$$= \ldots = 1$$

oder:

$$V = \int_0^1 \int_y^1 (3 - x - y)\,dx\,dy = \ldots = 1$$

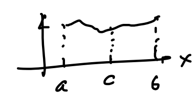
Eigenschaften von Doppelintegralen

(i) $\iint_\Omega c\,f(x,y)\,dA = c \iint_\Omega f(x,y)\,dA$ mit $c = \text{konst.}$

(ii) $\iint_\Omega \big(f(x,y) + g(x,y)\big)\,dA = \iint_\Omega f(x,y)\,dA + \iint_\Omega f(x,y)\,dA$

(iii) Sei $\Omega = \Omega_1 \cup \Omega_2$; $\Omega_1 \cap \Omega_2 \neq \{\}$

$$\longrightarrow \iint_\Omega f(x,y)\,dA = \iint_{\Omega_1} f(x,y)\,dA + \iint_{\Omega_2} f(x,y)\,dA \qquad \left(\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \right)$$



(iv) Sei $\Omega = R = [a,b] \times [c,d]$

$f(x,y) = g(x)\,h(y)$

$$\longrightarrow \iint_R f(x,y)\,dA = \iint_R g(x)\,h(y)\,dA = \left(\int_a^b g(x)\,dx \right)\left(\int_c^d h(y)\,dy \right)$$

$$\int_a^b \int_c^d g(x)\,h(y)\,dy\,dx = \int_a^b g(x) \boxed{\int_c^d h(y)\,dy}\,dx$$

$$= \left(\int_c^d h(y)\,dy \right)\left(\int_a^b g(x)\,dx \right)$$