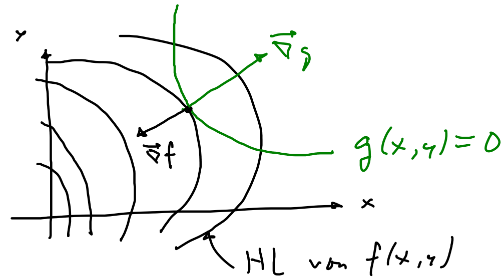
Lagrangemultiplikatoren

Lokale Min./Max. von $f(x,y)$

unter Bed. $g(x,y)=0$: $\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g = 0 \end{cases}$



Aufgabe: Ges: Min./Max. von $f(x,y) = xy$

auf Ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$

Lösg.: $f(x,y) = xy$

$g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1$; $\vec{\nabla} f = \begin{pmatrix} y \\ x \end{pmatrix}$; $\vec{\nabla}_g = \begin{pmatrix} \frac{x}{4} \\ y \end{pmatrix}$; $\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g = 0 \end{cases}$

$\longrightarrow \begin{cases} y = \frac{\lambda x}{4} & (1) \\ x = \lambda y & (2) \\ \frac{x^2}{8} + \frac{y^2}{2} = 1 & (3) \end{cases}$

$\frac{(1)}{(2)}$: $\frac{y}{x} = \frac{x}{4y} \longrightarrow 4y^2 = x^2$

$\longrightarrow x = \pm 2y$

in $(3) \longrightarrow \frac{4y^2}{8} + \frac{y^2}{2} = 1$

i.e. $y^2 = 1 \longrightarrow y = \pm 1$

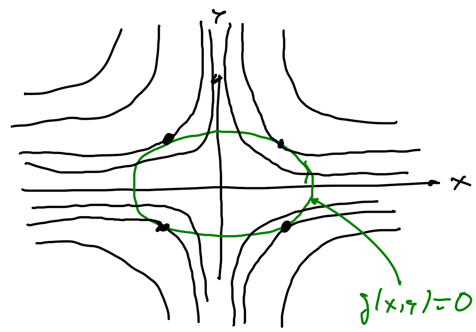
$\longrightarrow$ Pkte. für Extrema:

$(2,1) \longrightarrow f(2,1) = \underline{2}$ Max.

$(-2,1)$ $f(-2,1) = \underline{-2}$ Min.

$(2,-1)$ $f(2,-1) = \underline{-2}$ Min.

$(-2,-1)$ $f(-2,-1) = \underline{2}$ Max.



$f(x,y) = xy = C$

$C = 0 \rightarrow x = 0$ oder $y = 0$

$C = 1 \rightarrow y = \frac{1}{x}$

$C = -1 \rightarrow y = -\frac{1}{x}$

Aufgabe: Max./Min. von $f(x,y,z) = x - 2y + 5z$

auf Kugeloberfläche $x^2 + y^2 + z^2 = 30$

Lösg.: $\vec{\nabla} f = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$; $\vec{\nabla}_g = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$

$\longrightarrow \begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g = 0 \end{cases}$ ist: $\begin{cases} 1 = 2\lambda x & (1) \\ -2 = 2\lambda y & (2) \\ 5 = 2\lambda z & (3) \\ x^2 + y^2 + z^2 = 30 & (4) \end{cases}$

$-2\,(1)$: $-2 = -4\lambda x \overset{(2)}{=} 2\lambda y \longrightarrow -2x = y$

$5\,(1)$: $5 = 10\lambda x \overset{(3)}{=} 2\lambda z \longrightarrow 5x = z$

$\longrightarrow$ in (3) :

$x^2 + 4x^2 + 25x^2 = 30$

i.e. $30x^2 = 30$

$x^2 = 1 \rightarrow \underline{x = \pm 1}$

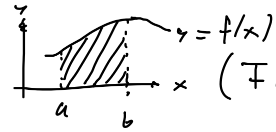
$\rightarrow y = \mp 2 \; ; \quad z = \pm 5$

$\rightarrow$ Pkte sind: $\underline{f(1,-2,5) = 30 \quad \text{Max.}}$

$\underline{f(-1,2,-5) = -30 \quad \text{Min.}}$

Mehrfachintegrale

Erweitern Konzept: $\int_a^b f(x)\,dx$

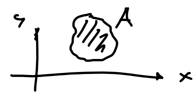


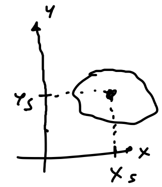
(Flächeninterpretation)

auf Integral von $f(x,y)$.

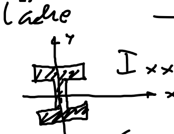
Anwendungen: (siehe später)

- Flächeninhalt





- Schwerpkt. einer Fläche
- Flächenmomente

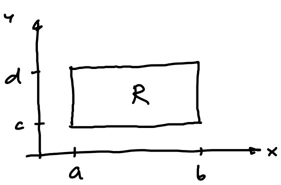


- Volumen / Masse eines Körpers
- Massenträgheitsmoment
- ....

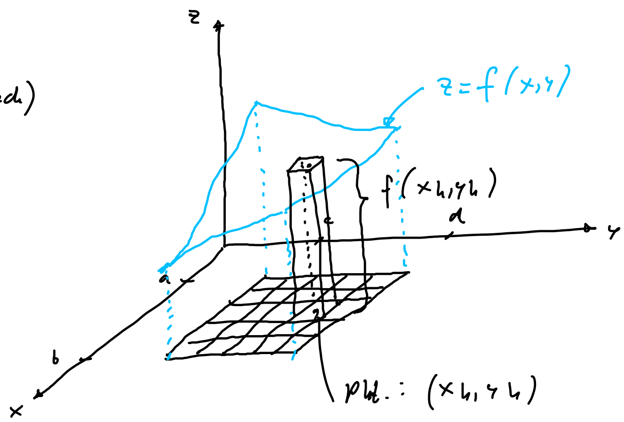
$$\left(\begin{array}{l} \vec{F} = m\vec{a} \\ \vec{M} = J\,\vec{\alpha} \end{array} \right)$$

Idee / Definition

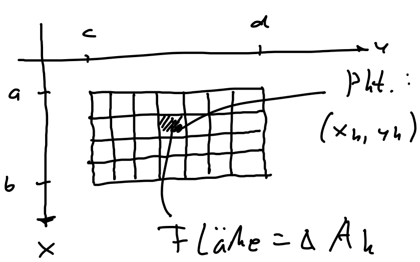
Betrachten $f(x,y)$ auf $R = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b \; ; \; c \leq y \leq d\}$



(Rechteck)



Einteilung von R in n Rechtecke:



Betrachten Summe: $S_n = \sum_{k=1}^{n} f(x_k, y_k) \cdot \Delta A_k$

(dies ist Approx. des Volumens zw. $z = f(x,y)$ & der x-y-Ebene).

Sei $\|P_n\| =$ max. Gittergrösse der Unterteilung von R

Def. Doppelintegral:

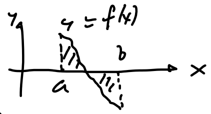
$$\boxed{\iint_R f(x,y)\,dA = \lim_{\substack{n \to \infty \\ \|P_n\| \to 0}} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k}$$

dA: Flächenelement

$f(x,y)\,dA = dV$: Volumenelement (mit Vorzeichen)

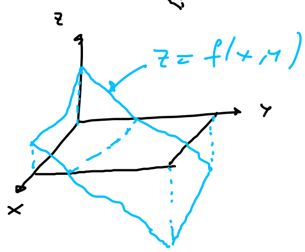
Doppelintegral hat (bis auf Vorzeichen) Interpretation eines Volumens ||

In 1D:



$$\int_a^b f(x)\,dx = 0$$ ist möglich!

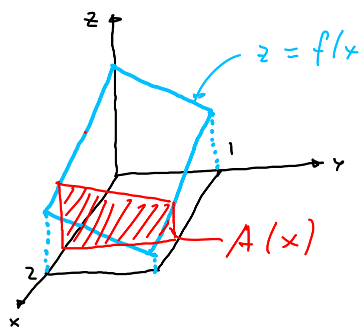
In 2D:



$$\iint_R f(x,y)\,dA = 0$$

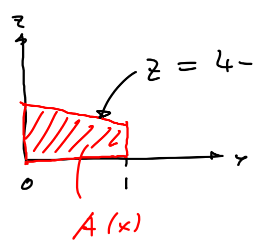
Berechnung: Illustration an Bsp: Integral von $f(x,y) = 4 - x - y$ in $R = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 2;\ 0 \leq y \leq 1\}$

Graph $z = f(x,y)$ ist Ebene:



$\longrightarrow$ $\iint_R f(x,y)\,dA$ entspricht Vol. zw. $z = 4 - x - y$ & $z = 0$ über R.

Idee: (i) Querschnittsfläche $A(x)$:



$$\longrightarrow A(x) = \int_{y=0}^{y=1} (4 - x - y)\,dy$$

(ii) Volumen aus Scheibenvolumen: $dV = A(x)\underbrace{dx}_{\text{Scheibendicke}}$

$$\longrightarrow V = \int_{x=0}^{x=2} A(x)\,dx$$

$$= \int_{x=0}^{x=2} \underbrace{\left(\int_{y=0}^{y=1} (4 - x - y)\,dy \right)}_{A(x)} dx$$

$$= \int_{x=0}^{x=2} \left(4y - xy - \frac{y^2}{2}\right)\Bigg|_{y=0}^{y=1} dx$$

innere Integration (betrachten x als Konst.)

$$= \int_{x=0}^{x=2} \left(\left(4 - x - \frac{1}{2}\right) - 0 \right) dx$$

$$= \int_{x=0}^{x=2} \left(\frac{7}{2} - x \right) dx = \left(\frac{7}{2} x - \frac{x^2}{2} \right) \Big|_{x=0}^{x=2}$$

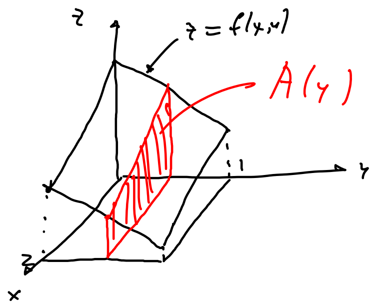
$$= (7 - 2) - 0 = \underline{\underline{5}}$$

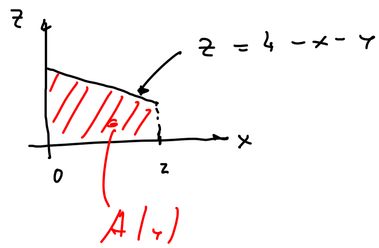
<u>äussere Integration</u>

I.e.:

$$\boxed{\iint_R f(x,y)\, dA = \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y)\, dy\, dx}$$

<u>Umkehrung Integrationsreihenfolge:</u>





$$\longrightarrow A(y) = \int_{x=0}^{x=2} (4 - x - y)\, dx$$

$$\longrightarrow V = \int_{y=0}^{y=1} A(y)\, dy$$

$$= \int_{y=0}^{y=1} \left(\int_{x=0}^{x=2} (4 - x - y)\, dx \right) dy$$

$$= \int_{y=0}^{y=1} \left(4x - \frac{x^2}{2} - xy \right) \Big|_{x=0}^{x=2} dy$$

$$= \int_{y=0}^{y=1} (6 - 2y)\, dy = (6y - y^2) \Big|_{y=0}^{y=1}$$

$$= 6 - 1 = \underline{\underline{5}}$$

$\longrightarrow$ Reihenfolge Integration spielt keine Rolle!

$$\longrightarrow \boxed{\iint_R f(x,y)\, dA = \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y)\, dy\, dx = \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y)\, dx\, dy}$$