Wiederholung:

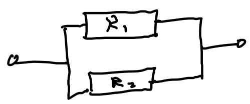
Tang.-Ebene an $z = f(x,y)$ im Pkt. $(x_0, y_0, \underset{f(x_0,y_0)}{\underset{\|}{z_0}})$:

$$z - z_0 = \underbrace{\left(\frac{\partial f}{\partial x}\right)_0}_{= \frac{\partial f}{\partial x}(x_0,y_0)} (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0)$$

Lin. Approx.: von $f(x,y)$ im Pkt. (x_0, y_0) :

$$f(x,y) \approx f(x_0,y_0) + \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0)$$

Aufgabe:



$$R = \frac{R_1 R_2}{R_1 + R_2}$$

$R_{10} = 10\,\Omega$
$R_{20} = 20\,\Omega$

Ges: Änderung von R, wenn: $\underset{\|}{\text{Änderung}}$ von R_1 : $+1\,\Omega$, R_2 : $-2\,\Omega$

$$\longrightarrow \Delta R \approx \underbrace{\left(\frac{\partial R}{\partial R_1}\right)_0}_{= \frac{\partial R}{\partial R_1}(R_{10}, R_{20})} \Delta R_1 + \left(\frac{\partial R}{\partial R_2}\right)_0 \Delta R_2$$

$$\frac{\partial R}{\partial R_1} = \ldots = \frac{R_2^2}{(R_1 + R_2)^2} \longrightarrow \left(\frac{\partial R}{\partial R_1}\right)_0 = \frac{4}{9}$$

$$\left(\frac{\partial R}{\partial R_2}\right)_0 = \frac{1}{9} \longrightarrow \underline{\Delta R} \approx \frac{4}{9} \cdot 1 + \frac{1}{9} \cdot (-2) = \underline{\frac{2}{9}}$$

Aufgabe: $P(U,R) = \frac{U^2}{R}$; Ges: $P(11, 99)$

$$P(U,R) \approx \underbrace{P(U_0, R_0)}_{= P(10,100) = 1} + \left(\frac{\partial P}{\partial U}\right)_0 (U - U_0) + \left(\frac{\partial P}{\partial R}\right)_0 (R - R_0)$$

wählen: $(U_0, R_0) = (10, 100)$

$$\frac{\partial P}{\partial U} = \frac{2U}{R} \longrightarrow \left(\frac{\partial P}{\partial U}\right)_0 = \frac{2 \cdot 10}{100} = \frac{1}{5}$$

$$\left(\frac{\partial P}{\partial R}\right)_0 = -\frac{1}{100} \longrightarrow P(11, 99) \approx 1 + \frac{1}{5} \cdot 1 - \frac{1}{100}(-1)$$

$$= \ldots = \frac{121}{100} = \underline{1{,}21}$$

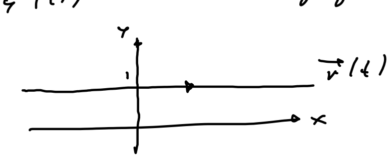
(Exakt. $1.\overline{2}$)

Kurve in x-y-Ebene ist Zuordnung: Intervall

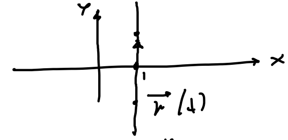
$$\vec{r} : I \longrightarrow \mathbb{R}^2$$
$$t \longmapsto \vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

(i.e. $x(t)$, $y(t)$: gegebene Fkt.)

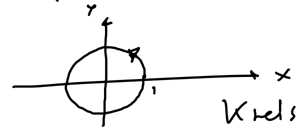
Bsp: (i) $\vec{r}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$



(ii) $\quad \vec{r}(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$

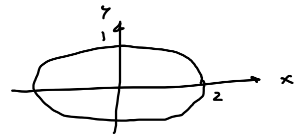


(iii) $\quad \vec{r}(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$



$$\left.\begin{array}{l} x(t) = \cos(t) \\ y(t) = \sin(t) \end{array}\right\} \longrightarrow x(t)^2 + y(t)^2 = 1$$

(iv) $\quad \vec{r}(t) = \begin{pmatrix} 2\cos(t) \\ \sin(t) \end{pmatrix}$

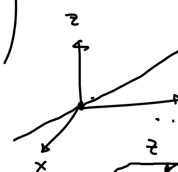


$$\left(\frac{x(t)^2}{2^2} + \frac{y(t)^2}{1^2} = 1 \right)$$

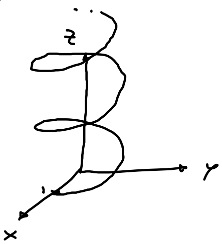
Analog: Kurven im Raum: $\quad \vec{r}: I \longrightarrow \mathbb{R}^3$

$$t \longmapsto \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

Bsp: (i) $\quad \vec{r}(t) = \begin{pmatrix} t \\ t \\ t \end{pmatrix}$



(iii) $\quad \vec{r}(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ t \end{pmatrix}$



Ableitung nach Parameter: Kettenregel

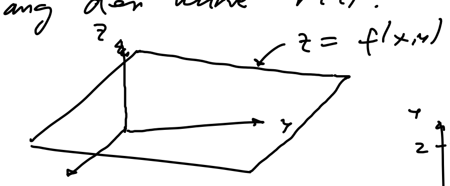
Seien gegeben: • Kurve in x-y-Ebene: $\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

• Fkt. von 2 Var.: $f(x,y)$

Hintereinanderschaltung ergibt: $\quad F: I \longrightarrow \mathbb{R}$

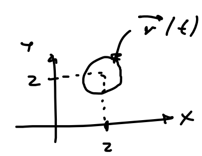
$$\begin{aligned} t \longmapsto z &= F(t) \\ &= f(\vec{r}(t)) \\ &= f(x(t), y(t)) \end{aligned}$$

$F(t)$ ist somit Fkt. $f(x,y)$ entlang der Kurve $\vec{r}(t)$.

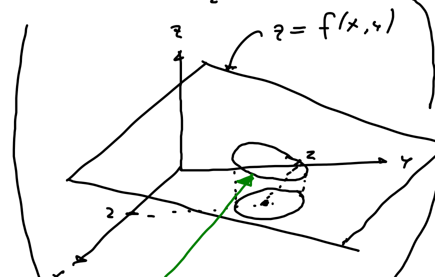


Bsp: $\quad f(x,y) = -\frac{1}{2}y + 4$

$$\vec{r}(t) = \begin{pmatrix} 2 + \cos(t) \\ 2 + \sin(t) \end{pmatrix}$$

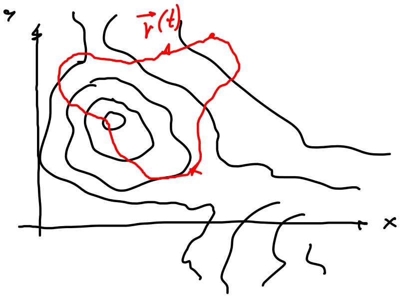


$$\longrightarrow \quad \underline{F(t)} = f(\vec{r}(t))$$
$$= f(2 + \cos(t), 2 + \sin(t))$$
$$= -\frac{1}{2}(2 + \sin(t)) + 4 = \underline{-\frac{1}{2}\sin(t) + 3}$$



$$\begin{pmatrix} x(t) \\ y(t) \\ F(t) \end{pmatrix} = \begin{pmatrix} 2 + \cos(t) \\ 2 + \sin(t) \\ -\frac{1}{2}\sin(t) + 3 \end{pmatrix}$$

Anwendung: (Wanderung)



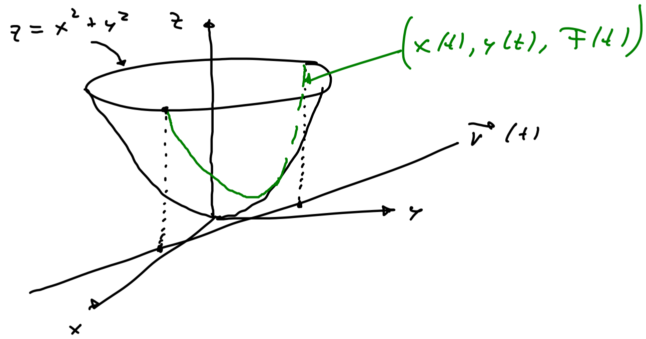
Höhe in Abh. der Zeit: $H(t) = h(\vec r(t))$

Aufgabe: $f(x,y) = x^2 + y^2$

$\vec r(t) = \begin{pmatrix} t \\ 1-t \end{pmatrix}$

Ges.: $F(t) = f(\vec r(t))$

Skizze ($z = f(x,y)$, $\vec r(t)$, ...)



Lösg.:

$$F(t) = f(\vec r(t)) = f(t, 1-t) = t^2 + (1-t)^2 = 2t^2 - 2t + 1$$

$\vec R(t) = \begin{pmatrix} x(t) \\ y(t) \\ F(t) \end{pmatrix}$ ist Kurve im Raum, die auf dem Graphen $z = f(x,y)$ und oberhalb von $\vec r(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ liegt.

Kettenregel: Sei $F(t) = f(x(t), y(t))$

$$\boxed{\frac{dF}{dt}(t) = \frac{\partial f}{\partial x}(x(t), y(t))\,\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(x(t), y(t))\,\frac{dy}{dt}(t)}$$

Erinnerung 1D:

$F(t) = f(g(t))$

$\longrightarrow \frac{dF}{dt}(t) = \frac{df}{dt}(g(t)) \cdot \frac{dg}{dt}(t)$

Oder, ohne Argumente:

$$\boxed{\frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$

Überprüfen an Bsp.:

$$\left.\begin{array}{l} f(x,y) = x^2 y + y^3 \\ \vec r(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t^2 \\ e^t \end{pmatrix} \end{array}\right\} F(t) = f(\vec r(t))$$

(i) Durch einsetzen:

$$F(t) = x(t)^2 y(t) + y(t)^3 = (t^2)^2 e^t + (e^t)^3 = t^4 e^t + e^{3t}$$

$$\longrightarrow \frac{dF}{dt}(t) = \underline{4t^3 e^t + t^4 e^t + 3e^{3t}}$$

(ii) Mit Kettenregel:

$$\frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= 2\underbrace{x}_{x(t)}\underbrace{y}_{y(t)} \cdot 2t + (x^2 + 3y^2) \cdot e^t$$

$$= 2t^2 e^t 2t + (t^4 + 3e^{2t}) e^t = \underline{4t^3 e^t + t^4 e^t + 3e^{3t}}$$

Aufgabe: Für $F(t) = f(\vec{r}(t))$

$f(x,y) = x^2 + y^2$

$\vec{r}(t) = \begin{pmatrix} t \\ 1-t \end{pmatrix}$

bestimme man $\frac{dF}{dt}(t)$

(i) mit Einsetzen

(ii) Kettenregel

Lösg: (i) $F(t) = x(t)^2 + y(t)^2$

$$= t^2 + (1-t)^2 = 2t^2 - 2t + 1$$

$$\longrightarrow \underline{\frac{dF}{dt}(t) = 4t - 2}$$

(ii) $\frac{\partial f}{\partial x} = 2x$; $\frac{\partial f}{\partial y} = 2y$; $\frac{dx}{dt} = 1$; $\frac{dy}{dt} = -1$

$$\longrightarrow \frac{dF}{dt} = 2\underset{x(t)=t}{\underset{\shortparallel}{x}} \cdot 1 + 2y \cdot (-1)$$

$$= 2t + 2(1-t)(-1) = \underline{4t - 2}$$