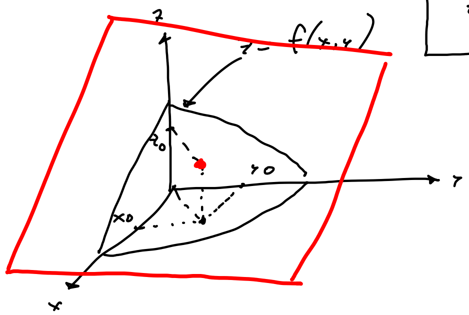
Wiederholung:

Gl. der Tang.-Ebene an $z = f(x,y)$ durch (x_0, y_0, z_0), wobei $z_0 = f(x_0, y_0)$:

$$\boxed{z - z_0 = \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0)}$$



Notation: $\left(\frac{\partial f}{\partial x}\right)_0 = \frac{\partial f}{\partial x}(x_0, y_0)$

Geschrieben mit

$\Delta z = z - z_0$

$\Delta x = x - x_0$

$\Delta y = y - y_0$

$$\longrightarrow \quad \underline{\Delta z = \left(\frac{\partial f}{\partial x}\right)_0 \Delta x + \left(\frac{\partial f}{\partial y}\right)_0 \Delta y}$$

Erinnerung: 1D:

$$\Delta y = \frac{df}{dx}(x_0)\,\Delta x$$

I.e.

$$\frac{\Delta y}{\Delta x} = \frac{df}{dx}(x_0)$$

Aufgabe:

In welchem Pkt. der Fläche $z = 2x^2 + y^2$ ist die Tang.-Ebene parallel zur Ebene $2x + 2y + 2z - 10 = 0$?

Lösg.: Idee: Normalenvektoren müssen vielfache voneinander sein:

Fläche: $\vec{n}_1 = \begin{pmatrix} \left(\frac{\partial f}{\partial x}\right)_0 \\ \left(\frac{\partial f}{\partial y}\right)_0 \\ -1 \end{pmatrix}$ Ebene: $\vec{n}_2 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

$\longrightarrow \boxed{\vec{n}_1 = h\,\vec{n}_2}$

$\frac{\partial f}{\partial x} = 4x$; $\frac{\partial f}{\partial y} = 2y \longrightarrow \vec{n}_1 = \begin{pmatrix} 4x \\ 2y \\ -1 \end{pmatrix}$

$$\longrightarrow \begin{pmatrix} 4x_0 \\ 2y_0 \\ -1 \end{pmatrix} = h \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \longrightarrow h = -\frac{1}{2}$$

$$\left.\begin{array}{l} x_0 = -\frac{1}{4} \\ y_0 = -\frac{1}{2} \end{array}\right\} \longrightarrow z_0 = f(x_0, y_0) = 2\left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{2}\right)^2 = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$$

$$\longrightarrow \underline{(x_0, y_0, z_0) = \left(-\frac{1}{4}, -\frac{1}{2}, \frac{3}{8}\right)}$$

Alternative:

Ebene: $z = -x - y + 10$

Idee: Steigungen sind gleich:

$$\longrightarrow \left.\begin{array}{l} \frac{\partial g}{\partial x} = \boxed{-1} \quad \Big| \quad \frac{\partial f}{\partial x} = \boxed{4x} \longrightarrow \underline{x_0 = -\frac{1}{4}} \\ \frac{\partial g}{\partial y} = \boxed{-1} \quad \Big| \quad \frac{\partial f}{\partial y} = \boxed{2y} \longrightarrow \underline{y_0 = -\frac{1}{2}} \end{array}\right\} \longrightarrow \underline{z_0} = \ldots = \underline{\frac{3}{8}}$$

Lineare Approximation:

Erinnerung: 1D-Fall:

$$f(x) \approx f(x_0) + \frac{df}{dx}(x_0)(x - x_0)$$

ist lin. Approx. von $f(x)$ an Stelle x_0.

Für $f(x,y)$: $\quad f(x,y) \approx f(x_0,y_0) + \left(\frac{\partial f}{\partial x}\right)_0 (x-x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y-y_0)$

ist lin. Approx. von $f(x,y)$ an Stelle (x_0, y_0).

Aufgabe: Man linearisiere $f(x,y) = x^2 + y^4 + e^{xy}$

an Stelle $(x_0, y_0) = (1, 0)$.

Lösg.:

$f(x_0,y_0) = 2$

$\frac{\partial f}{\partial x} = 2x + y e^{xy} \longrightarrow \left(\frac{\partial f}{\partial x}\right)_0 = 2$

$\frac{\partial f}{\partial y} = 4y^3 + x e^{xy} \longrightarrow \left(\frac{\partial f}{\partial y}\right)_0 = 1$

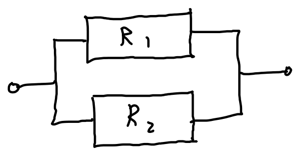
$\longrightarrow$ Linearisierung:

$f(x,y) \approx f(x_0,y_0) + \left(\frac{\partial f}{\partial x}\right)_0 (x-x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y-y_0)$

$= 2 + 2(x-1) + 1 \cdot (y-0)$

$= \underline{2x + y}$

Aufgabe: In der Schaltung



$$R(R_1, R_2) = \frac{R_1 R_2}{R_1 + R_2}$$

sind $R_1 = 10\,\Omega$, $R_2 = 20\,\Omega$.
Wie ändert sich (approximativ) der Gesamtwiderstand,
wenn sich R_1 um $+1\,\Omega$ und R_2 um $-2\,\Omega$ ändert?
Man benutze lin. Approx.

Lösg.: Benutzen: $\Delta R \approx \left(\frac{\partial R}{\partial R_1}\right)_0 \Delta R_1 + \left(\frac{\partial R}{\partial R_2}\right)_0 \Delta R_2$

$$\frac{\partial R}{\partial R_1} = \frac{R_2}{R_1+R_2} - \frac{R_1 R_2}{(R_1+R_2)^2} = \frac{R_2^2}{(R_1+R_2)^2} \longrightarrow \left(\frac{\partial R}{\partial R_1}\right)_0 = \frac{\partial R}{\partial R_1}(\underset{10}{R_{10}}, \underset{20}{R_{20}}) = \frac{20^2}{(10+20)^2} = \frac{4}{9}$$

$$\frac{\partial R}{\partial R_2} = \; \ldots\ldots \; = \frac{R_1^2}{(R_1+R_2)^2} \longrightarrow \left(\frac{\partial R}{\partial R_2}\right)_0 = \frac{\partial R}{\partial R_2}(\underset{10}{R_{10}}, \underset{20}{R_{20}}) = \frac{10^2}{(10+20)^2} = \frac{1}{9}$$

$$\longrightarrow \underline{\Delta R} \approx \frac{4}{9} \cdot \Delta R_1 + \frac{1}{9} \Delta R_2 = \frac{4}{9} \cdot 1 + \frac{1}{9} \cdot (-2) = \underline{\frac{2}{9}}$$

$\Delta R_1 = 1$

$\Delta R_2 = -2$

Aufgabe: $P(U,R) = \frac{U^2}{R}$ $\quad$ Ges: $P(11, 99)$ mit lin. Approx.

Lösg.: $P(U,R) \approx P(U_0,R_0) + \left(\frac{\partial P}{\partial U}\right)_0 (U-U_0) + \left(\frac{\partial P}{\partial R}\right)_0 (R-R_0)$

wählen: $(U_0, R_0) = (10, 100)$

$P(U_0,R_0) = \frac{10^2}{100} = 1$

$\frac{\partial P}{\partial U} = \frac{2U}{R} \longrightarrow \left(\frac{\partial P}{\partial U}\right)_0 = \frac{2 \cdot 10}{100} = \frac{1}{5}$

$\frac{\partial P}{\partial R} = -\frac{U^2}{R^2} \longrightarrow \left(\frac{\partial P}{\partial R}\right)_0 = -\frac{10^2}{100^2} = -\frac{1}{100}$

$\longrightarrow \underline{P(11,99)} \approx 1 + \frac{1}{5}(11-10) - \frac{1}{100}(99-100)$

$= 1 + \frac{1}{5} + \frac{1}{100}$

$= \frac{100+20+1}{100} = \frac{121}{100}$

$= \underline{1{,}21}$

(Exakt: $P(11,99) = \frac{11^2}{99} = 1.\overline{2}$)