Part. Ableitungen höherer Ordnung

$$\begin{tikzcd}[column sep=small]
 & & f(x,y) \arrow[dl, bend right, "\frac{\partial}{\partial x}"'] \arrow[dr, bend left, "\frac{\partial}{\partial y}"] & & \\
 & \frac{\partial f}{\partial x}(x,y) \arrow[dl, bend right, "\frac{\partial}{\partial x}"'] \arrow[d, bend left, "\frac{\partial}{\partial y}"] & & \frac{\partial f}{\partial y}(x,y) \arrow[d, bend right, "\frac{\partial}{\partial x}"'] \arrow[dr, bend left, "\frac{\partial}{\partial y}"] & \\
\frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial y \partial x}(x,y) & & \frac{\partial^2 f}{\partial x \partial y}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y)
\end{tikzcd}$$

Aufgabe: Man bestimme: $\frac{\partial^2 f}{\partial x^2}$; $\frac{\partial^2 f}{\partial y \partial x}$; $\frac{\partial^2 f}{\partial x \partial y}$; $\frac{\partial^2 f}{\partial y^2}$

für $f(x,y) = x \cos(y) + y e^x$

Lösg.:

$$\frac{\partial f}{\partial x} = \cos(y) + y e^x$$

$$\frac{\partial^2 f}{\partial x^2} = y e^x \quad ; \quad \underline{\frac{\partial^2 f}{\partial y \partial x} = -\sin(y) + e^x}$$

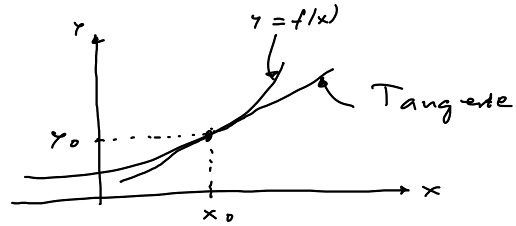
$$\frac{\partial f}{\partial y} = -x \sin(y) + e^x$$

$$\underline{\frac{\partial^2 f}{\partial x \partial y} = -\sin(y) + e^x} \quad ; \quad \frac{\partial^2 f}{\partial y^2} = -x \cos(y).$$

Allgemein: Reihenfolge bei partiellen Ableitungen darf vertauscht werden.

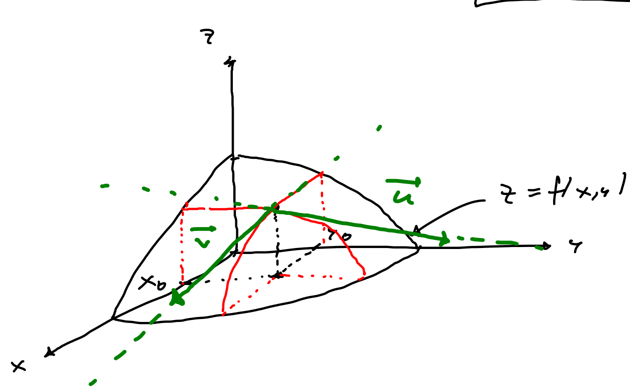
Tangentialebene

Erinnerung: 1D-Fall: Tangente:



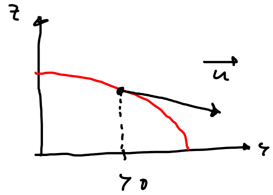
Gl. Tang. an $y = f(x)$ in (x_0, y_0)
(wobei $y_0 = f(x_0)$) : $\boxed{y - y_0 = \frac{df}{dx}(x_0)(x - x_0)}$

Tang.-Ebene:

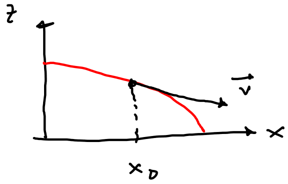


Ges: Gl. der Tang.-Ebene an $z = f(x,y)$ durch Pkt. $(x_0, y_0, f(x_0, y_0))$

Idee: Normalenvektor aus Kreuzprodukt von $\vec{u}, \vec{v}$.



$\longrightarrow \vec{u} = \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y}(x_0,y_0) \end{pmatrix}$



$\longrightarrow \vec{v} = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x}(x_0,y_0) \end{pmatrix}$

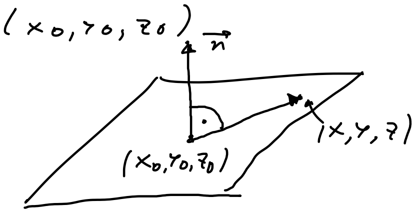
$$\left.\begin{matrix} \vec{u} \\ \vec{v} \end{matrix}\right\} \vec{n} = \vec{u} \times \vec{v} = \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y} \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ -1 \end{pmatrix}$$

I.e. $$\vec{n} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0,y_0) \\ \frac{\partial f}{\partial y}(x_0,y_0) \\ -1 \end{pmatrix}$$

Ebenengl. mit Normalenvektor:

Ebene mit Normalenvektor $\vec{n}$ durch Pkt. (x_0,y_0,z_0) ist geg. durch:

$$\vec{n} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = 0$$



Mit: $\vec{n} = \begin{pmatrix} \left(\frac{\partial f}{\partial x}\right)_0 \\ \left(\frac{\partial f}{\partial y}\right)_0 \\ -1 \end{pmatrix}$

(Notation: $\left(\frac{\partial f}{\partial x}\right)_0 = \frac{\partial f}{\partial x}(x_0,y_0)$)

folgt: <u>Gl. der Tang.-Ebene</u> an $z = f(x,y)$ durch Pkt. (x_0,y_0,z_0)
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad f(x_0,y_0)$:

$$\left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0) - (z - z_0) = 0$$

Bem. Dies ist kartesische Form: $ax + by + cz = d$

mit:

$$a = \left(\frac{\partial f}{\partial x}\right)_0$$
$$b = \left(\frac{\partial f}{\partial y}\right)_0$$
$$c = -1$$
$$d = x_0\left(\frac{\partial f}{\partial x}\right)_0 + y_0\left(\frac{\partial f}{\partial y}\right)_0 - z_0$$

Umstellen:

$$\boxed{z - z_0 = \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0)}$$

Dies ist analog zu Gl. im 1D-Fall:

$$y - y_0 = \frac{df}{dx}(x_0)(x - x_0)$$

Bsp.: Gl. Tang.-Ebene an $z = f(x,y) = x^2 + y^2 + e^{xy}$
im Pkt. $(1,0,2)$
└ müsste nicht geg. sein.

$(x_0, y_0, z_0) = (1, 0, 2)$

$\frac{\partial f}{\partial x} = 2x + y e^{xy}$; $\frac{\partial f}{\partial y} = 2y + x e^{xy}$

$\left(\frac{\partial f}{\partial x}\right)_0 = \frac{\partial f}{\partial x}(x_0, y_0, z_0) = \frac{\partial f}{\partial x}(1, 0, 2) = 2 \cdot 1 + 0 \cdot e^{1 \cdot 0} = 2$

$\left(\frac{\partial f}{\partial y}\right)_0 = \dots = 2 \cdot 0 + 1 \cdot e^{1 \cdot 0} = 1$

$\longrightarrow$ Gl. ist: $z - z_0 = \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0)$

$\longrightarrow$ $z - 2 = 2(x - 1) + 1 \cdot (y - 0)$

I.e. $z - 2 = 2x - 2 + y$

I.e. $z = 2x + y$

Aufgabe: Ges: Gl. Tang.-Ebene an $z = f(x,y) = x^2 - 3xy$
im Pkt. $(1, 2, *)$.

Lösg.: $z_0 = f(x_0, y_0) = 1^2 - 3 \cdot 1 \cdot 2 = -5 \longrightarrow (x_0, y_0, z_0) = (1, 2, -5)$

$\frac{\partial f}{\partial x} = 2x - 3y \longrightarrow \left(\frac{\partial f}{\partial x}\right)_0 = 2 \cdot 1 - 3 \cdot 2 = -4$

$\frac{\partial f}{\partial y} = -3x \longrightarrow \left(\frac{\partial f}{\partial y}\right)_0 = -3 \cdot 1 = -3$

$\longrightarrow$ Gl. ist: $z - z_0 = \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0)$

$\longrightarrow$ $z + 5 = -4(x - 1) - 3(y - 2)$

I.e. $z = -4x - 3y + 5$