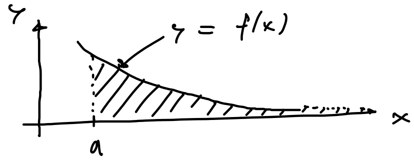
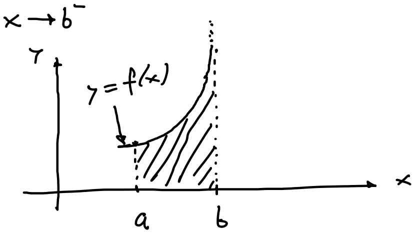
Uneigentliche Integrale

sind Integrale der Form: (i) $\int_a^\infty f(x)\,dx$



(ii) $\int_a^b f(x)\,dx$ mit $\lim_{x\to b^-} |f(x)| = \infty$



Def: (i) $\underline{\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx}$

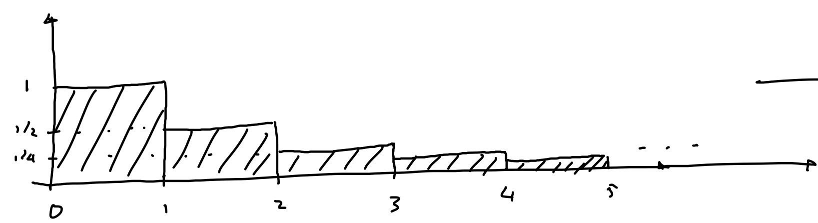
(ii) Für $f(x)$ mit $\lim_{x\to b^-} |f(x)| = \infty$

$$\underline{\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx}$$

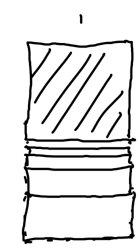
Falls Grenzwerte ex. heissen Integrale <u>konvergent</u>, ansonsten <u>divergent</u>.

Bsp: $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left(-\frac{1}{x}\right)\Big|_1^b$

$= \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right) = \underline{1}$ (konvergent)



$\longrightarrow \sum$ Flächen =



= 2

Bsp: $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \operatorname{Log}(x)\Big|_1^b$

$= \lim_{b\to\infty} \operatorname{Log}(b) - \underbrace{\operatorname{Log}(1)}_{=0}$

$= \lim_{b\to\infty} \operatorname{Log}(b) = \underline{\infty}$ divergent!

Aufgabe: Ges: Volumen Rotationskörper, der entsteht wenn man $y = \frac{1}{x}$; $x \in [1, \infty)$ um x-Achse rotiert.

Lösg: $V = \pi \int_1^\infty f(x)^2\,dx = \pi \int_1^\infty \frac{1}{x^2}\,dx$

$= \pi \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \pi \lim_{b\to\infty} \left(1 - \frac{1}{b}\right) = \underline{\pi}$

Bsp:

$$\int_0^1 \frac{1}{\sqrt{1-x}}\,dx = \lim_{b\to 1^-} \int_0^b \frac{1}{\sqrt{1-x}}\,dx$$

$$= \lim_{b\to 1^-} -2(1-x)^{1/2}\Big|_0^b$$

$$= \lim_{b\to 1^-} -2\left(\sqrt{1-b}-1\right) = \underline{2}$$

Aufgabe: Ges: $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

Hinweise: $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$

$$\int_{-\infty}^{\infty} .. = \int_{-\infty}^{a} \dots + \int_{a}^{\infty} \dots$$

Lösg.:

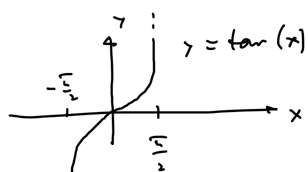
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^{\infty} \frac{1}{1+x^2}\,dx$$

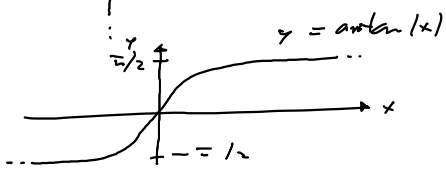
$$= \lim_{b\to -\infty} \int_b^0 \frac{1}{1+x^2}\,dx + \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\,dx$$

$$= \lim_{b\to -\infty} \arctan(x)\Big|_b^0 + \lim_{b\to\infty} \arctan(x)\Big|_0^b$$

$$= \lim_{b\to -\infty} \left(0 - \arctan(b)\right) + \lim_{b\to\infty} \left(\arctan(b) - 0\right)$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \underline{\pi}$$





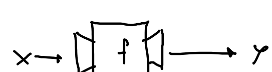
$$\int_a^{\infty} \dots = \dots\Big|_a^{\infty} = \dots$$

Material 1. Prüfung

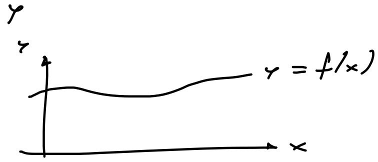
Funktionen von mehreren Variablen

Bis jetzt

$$f: D \longrightarrow \mathbb{R}$$
$$x \longmapsto y = f(x)$$



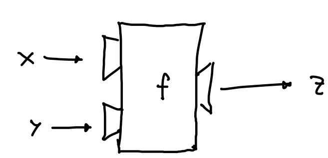
Darstellung:



D: Def.-Bereich, $D \subset \mathbb{R}$.

Funktionen von zwei Variablen

$$f: D \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto z = f(x,y)$$



D: Def.-Bereich, $\underline{\underline{D \subset \mathbb{R}^2}}$

Erinnerung: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ (kartesisches Produkt)

$= \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}\}$

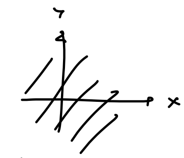
$= $ Menge aller geordneten Paare (x,y)

$(a,b) \neq (b,a)$

$\{a,b\} = \{b,a\}$

$= $ —— " —— 2-Tupel

$= $ —— " —— Punkte in x-y-Ebene



$\longrightarrow$ Def.-Bereich ist Teilmenge der x-y-Ebene!

Bsp:

(i) $U = f(R, I) = RI$

(ii) $f(x,y) = \log(x+y)$

(iii) $V = f(r,h) = r^2 \pi h$ (Vol. Zylinder)

Aufgabe: $f(x,y) = \sin(xy)$

Ges: $f\left(2, \frac{\pi}{6}\right)$

$f\left(-\frac{\pi}{2}, -7\right)$

Lösg:

$f\left(2, \frac{\pi}{6}\right) = \sin\left(2 \cdot \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$f\left(-\frac{\pi}{2}, -7\right) = \sin\left(\left(-\frac{\pi}{2}\right)(-7)\right) = \sin\left(7 \cdot \frac{\pi}{2}\right)$

$= -1$

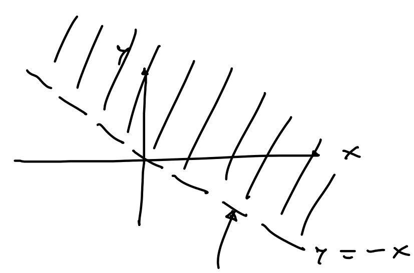
Aufgabe: Man bestimme & skizziere Def.-Bereich für:

(i) $f(x,y) = \log(x+y)$

(ii) $g(x,y) = \sqrt{y - x^2}$

Lösg: (i) $x + y > 0$

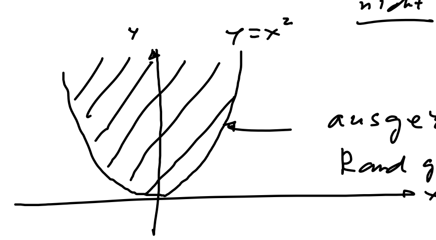
$D = \{(x,y) \mid \underbrace{x+y>0}_{y > -x}\}$



gestrichelt: Rand gehört nicht dazu!

(ii) $y - x^2 \geq 0 \longrightarrow y \geq x^2$

$D = \{(x,y) \mid y \geq x^2\}$



ausgezogen: Rand gehört dazu!

Bsp: Def.- & Wertebereich von $f(x,y) = \log(36 - 4x^2 - 9y^2)$

Def.-Bereich: $36 - 4x^2 - 9y^2 > 0$

$4x^2 + 9y^2 < 36 \quad / : 36$

$\frac{x^2}{9} + \frac{y^2}{4} < 1$

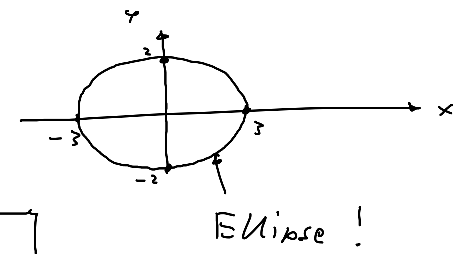
I.e. $D = \{ (x,y) \mid \frac{x^2}{9} + \frac{y^2}{4} < 1 \}$

Wie sieht dieses Gebiet aus?

Idee: betrachten: $\frac{x^2}{9} + \frac{y^2}{4} = 1$ (ohne 9, 4 wäre das Kreis mit Radius 1 am Ursprung).

$x = 0 \longrightarrow \frac{y^2}{4} = 1 \longrightarrow y = \pm 2$

$y = 0 \longrightarrow \frac{x^2}{9} = 1 \longrightarrow x = \pm 3$



Allg.: $\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ ist Ellipse mit Halbachsen } a \,\&\, b.}$