Wiederholung:

Substitution: **Bsp:**

$$\int \frac{6x^2}{(1-4x^3)^3}\,dx \underset{\substack{\uparrow \\ u = 1-4x^3}}{=} -\int \frac{6x^2}{u^3}\,\frac{du}{12x^2} = -\frac{1}{2}\int u^{-3}\,du$$

$$\to \frac{du}{dx} = -12x^2$$

$$\to dx = -\frac{du}{12x^2}$$

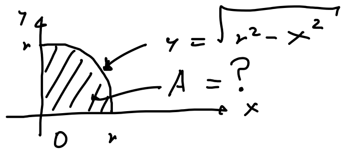
$$= -\frac{1}{2}\cdot\frac{u^{-2}}{-2} + C$$

$$= \frac{1}{4}u^{-2} + C$$

$$\underset{\substack{\uparrow \\ u = 1-4x^3}}{=} \underline{\frac{1}{4}\cdot\frac{1}{(1-4x^3)^2} + C}$$

$$\left[\ \text{Allg. Form:} \quad \int \underbrace{\underbrace{\phi(g(x))}_{\text{verschachtelte}}\ \overbrace{\frac{dg}{dx}}^{\text{innere Ableitung}}(x)}_{\text{Fkt.} \ = \frac{d}{dx}\Phi(g(x))}\,dx = \Phi(g(x)) + C \ \right]$$

Kreisfläche: (Subst. mit Grenzen)



$$\to A = \int_0^r \sqrt{r^2 - x^2}\,dx$$

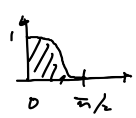
$$\underset{\substack{\uparrow \\ x = r\sin(u)}}{=} \int_0^{\pi/2} \sqrt{r^2 - r^2\sin^2(u)}\; r\cos(u)\,du = r^2 \underbrace{\int_0^{\pi/2} \cos^2(u)\,du}_{?}$$

$$dx = r\cos(u)\,du$$

$$x = 0 \to \underline{u = 0}$$

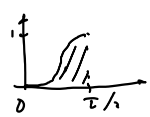
$$x = r \to \underline{u = \frac{\pi}{2}}$$

$$\int_0^{\pi/2} \cos^2(u)\,du$$



$$\longrightarrow \int_0^{\pi/2} \underbrace{\left(\cos^2(u) + \sin^2(u)\right)}_{=1} du = \int_0^{\pi/2} du = \frac{\pi}{2}$$

$$\int_0^{\pi/2} \sin^2(u)\,du$$



$$\longrightarrow \int_0^{\pi/2} \cos^2(u)\,du = \underline{\frac{\pi}{4}}$$

$$\longrightarrow \underline{A} = r^2 \int_0^{\pi/2} \cos^2(u)\,du = \underline{\frac{r^2\pi}{4}}$$

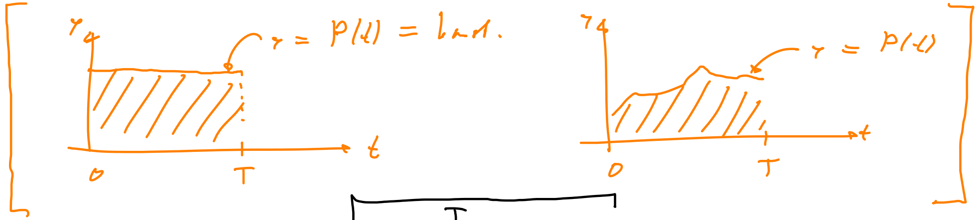
Bsp: (Elektrotechnik)

Frage: Wechselstrom $I(t) = I_0 \sin(\omega t)$ liegt an Ohm'schem Widerstand. Welcher Gleichstrom erzeugt im Mittel die selbe Leistung?

Leistung: $P = U \cdot I = I^2 R$

Arbeit in einer Periode T: (i) Gleichstrom: $W = \boxed{PT} = I_{eff}^2 R T$ $\quad (*)$

(ii) Wechselstrom: $W = \int_0^T P(t)\,dt = \int_0^T I(t)^2 R\,dt$ $\quad (**)$



Benötigen: $(*) = (**)$ $\longrightarrow$ $I_{eff} = \sqrt{\frac{1}{T} \underbrace{\int_0^T I(t)^2 dt}_{= ?}}$

$$\int_0^T I(t)^2 dt = I_0^2 \int_0^T \sin^2(\omega t)\, dt \qquad\qquad \int_0^{\pi/2} \sin^2(u)\, du = \frac{\pi}{4}$$

$$= \frac{I_0^2}{\omega} \int_0^{2\pi} \sin^2(u)\, du = \frac{I_0^2}{\omega}\pi = \frac{I_0^2 \pi T}{2\pi} = \frac{I_0^2 T}{2}$$

$u = \omega t$; $t = 0 \rightarrow u = 0$

$t = T \rightarrow u = \omega T = \frac{2\pi}{T} T = 2\pi$

$\frac{du}{dt} = \omega$

$dt = \frac{du}{\omega}$

$$\longrightarrow \underline{\underline{I_{eff}}} = \sqrt{\frac{1}{T} \cdot \frac{I_0^2 T}{2}} = \underline{\underline{\frac{I_0}{\sqrt{2}}}}$$

I_{eff} nennt man auch RMS – Wert $\sqrt{\frac{1}{T}\int_0^T I(t)^2 dt}$

(von root – mean – square)

Allgemein definiert man: Quadratischen Mittelwert von $f(t)$:

$$\boxed{f_{rms} = \sqrt{\frac{1}{T}\int_0^T f(t)^2 dt}}$$

Partielle Integration:

Idee: Produktregel: $\frac{d}{dx}\left(f(x) g(x)\right) = \frac{df}{dx}(x)\, g(x) + f(x) \frac{dg}{dx}(x)$

$$\longrightarrow f(x) \frac{dg}{dx}(x) = \frac{d}{dx}\left(f(x) g(x)\right) - \frac{df}{dx}(x)\, g(x)$$

integrieren:

$$\int f(x) \frac{dg}{dx}(x) dx = f(x) g(x) - \int \frac{df}{dx}(x)\, g(x) dx$$

Mit '-Notation: $\boxed{\int f(x) \cdot g'(x) dx = f(x) g(x) - \int f'(x) g(x)\, dx}$

Formel für partielle Int.

$$\left(\begin{aligned} &(fg)' = f'g + fg' \\ &\longrightarrow \int fg' = \underbrace{\int (fg)'}_{fg} - \int f'g \end{aligned} \right)$$

Man merke sich:

$$\boxed{\int fg' = fg - \int f'g}$$

Bsp:

$$\int x\cos(x)\,dx = x\sin(x) - \int 1\cdot\sin(x)\,dx = \underline{x\sin(x) + \cos(x) + C}$$

$$\int f(x)\, g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$f(x) = x \longrightarrow f'(x) = 1$$

$$g'(x) = \cos(x) \longrightarrow g(x) = \sin(x)$$

Bsp:

$$\int \underset{f}{x}\,\underset{g'}{e^x}\,dx \overset{\int fg' = fg - \int f'g}{=} xe^x - \int e^x\,dx = xe^x - e^x + C = \underline{e^x(x-1) + C}$$

$$f(x) = x \longrightarrow f'(x) = 1$$

$$g'(x) = e^x \longrightarrow g(x) = e^x$$

Bsp:

$$\int \underset{f}{x^2}\,\underset{g'}{e^x}\,dx \overset{\int fg' = fg - \int f'g}{=} x^2e^x - \int 2xe^x\,dx$$

$$= x^2e^x - 2\int \underset{f}{x}\,\underset{g'}{e^x}\,dx$$

nochmals part. int.

$$= x^2e^x - 2\left(xe^x - \int e^x\,dx\right)$$

$$= x^2e^x - 2\left(xe^x - e^x\right) + C$$

$$= \underline{e^x\left(x^2 - 2x + 2\right) + C}$$

Bsp:

$$\int \underbrace{\log(x)}_{=f(x)}\,dx \overset{\int fg' = fg - \int f'g}{=} x\log(x) - \int \frac{1}{x}\,x\,dx = x\log(x) - \int dx$$

$$;\ g'(x) = 1$$

$$= x\log(x) - x + C$$

$$f(x) = \log(x) \longrightarrow f'(x) = \frac{1}{x}$$

$$g'(x) = 1 \longrightarrow g(x) = x$$

$$= \underline{x(\log(x) - 1) + C}$$

Aufgabe: $\int x\sin(x)\,dx = ?$

Lösg:

$$\int \underset{f}{x}\,\underset{g'}{\sin(x)}\,dx \overset{\int fg' = fg - \int f'g}{=} -x\cos(x) + \int \cos(x)\,dx$$

$$= \underline{-x\cos(x) + \sin(x) + C}$$

Bsp:

$$\int \sin^2(x)\,dx = -\sin(x)\cos(x) + \int \cos^2(x)\,dx$$

$$f(x) = \sin(x) \longrightarrow f'(x) = \cos(x)$$

$$g'(x) = \sin(x) \longrightarrow g(x) = -\cos(x)$$

$$= -\sin(x)\cos(x) + \int (1-\sin^2(x))\,dx$$

$$= \underline{-\sin(x)\cos(x) + x - \int \sin^2(x)\,dx} \quad \Big/ + \int \sin^2(x)\,dx$$

$$\longrightarrow \quad 2\int \sin^2(x)\,dx = -\sin(x)\cos(x) + x$$

$$\longrightarrow \quad \underline{\int \sin^2(x)\,dx = \frac{1}{2}\left(-\sin(x)\cos(x) + x\right) + C}$$

Partielle Integration mit Grenzen:

Es gilt: $$\underline{\int_a^b f(x)g'(x)\,dx = f(x)\,g(x)\Big|_a^b - \int_a^b f'(x)\,g(x)\,dx}$$

Bsp.: $$\int_0^2 \underset{f}{x}\,\underset{g'}{e^x}\,dx = x e^x\Big|_0^2 - \int_0^2 e^x\,dx$$

$$= 2e^2 - 0 - e^x\Big|_0^2$$

$$= 2e^2 - (e^2 - 1) = 2e^2 - e^2 + 1 = \underline{e^2 + 1}$$

Aufgabe:

(i) $\int x^2 \log(x)\,dx$

(ii) $\int x\sqrt{1+x}\,dx$

(iii) $\int x^2 \sin(x)\,dx$

Lösg: (i) $$\int x^2 \log(x)\,dx = \log(x)\cdot\frac{x^3}{3} - \int \frac{1}{x}\cdot\frac{x^3}{3}\,dx$$

$f(x) = \log(x)$

$g'(x) = x^2$

$$= \frac{x^3}{3}\log(x) - \frac{1}{3}\int x^2\,dx$$

$$= \underline{\frac{x^3}{3}\left(\log(x) - \frac{1}{3}\right) + C}$$

(ii) $$\int x\sqrt{1+x}\,dx = x\,\frac{2}{3}(1+x)^{\frac{3}{2}} - \frac{2}{3}\int (1+x)^{\frac{3}{2}}\,dx$$

$f(x) = x$

$g'(x) = \sqrt{1+x}$

$$= \underline{\frac{2}{3}x(1+x)^{\frac{3}{2}} - \frac{4}{15}(1+x)^{5/2} + C}$$

(iii) $$\int x^2 \sin(x)\,dx = -x^2\cos(x) + \int 2x\cos(x)\,dx$$

$$= -x^2\cos(x) + 2\left(x\sin(x) - \int \sin(x)\,dx\right)$$

$$= \underline{-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C}$$