Substitutionsmethode

Formales Vorgehen: Beispiele

(i) $\int 2x\cos(x^2)\,dx$ Idee: Substitution: $u = x^2$

$$\longrightarrow \frac{du}{dx} = 2x$$

$$\longrightarrow dx = \frac{du}{2x}$$

$$\longrightarrow \int 2x\cos(x^2)\,dx = \int \cancel{2x}\cos(u)\frac{du}{\cancel{2x}}$$

$$= \int \cos(u)\,du = \sin(u) + C$$

$$\underset{\text{Rücksubstitution } (u = x^2)}{=} \sin(x^2) + C$$

(ii) $$\int e^{3x}\,dx \underset{\substack{u = 3x \\ \frac{du}{dx} = 3 \\ dx = \frac{du}{3}}}{=} \int e^u \frac{du}{3} = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C$$

$$\underset{\text{Rücksubstitution } (u = 3x).}{=} \frac{1}{3}e^{3x} + C$$

Allgemein: Betrachten Integrale der Form

$$\int f(x)\,dx = \int \phi(g(x))\frac{dg}{dx}(x)\,dx \quad ,$$

wobei wir von ϕ die Stammfkt. kennen. I.e. wir kennen $\Phi(x)$, so dass: $\frac{d\Phi}{dx}(x) = \phi(x)$

$$\longrightarrow \frac{d\Phi}{dx}(g(x)) = \phi(g(x))$$

$$\longrightarrow \frac{d}{dx}\Phi(g(x)) \underset{\text{Kettenregel}}{=} \frac{d\Phi}{dx}(g(x))\frac{dg}{dx}(x) = \phi(g(x))\frac{dg}{dx}(x)$$

$$\longrightarrow \int f(x)\,dx = \int \phi(g(x))\frac{dg}{dx}(x)\,dx = \int\left(\frac{d}{dx}\Phi(g(x))\right)dx$$

$$= \Phi(g(x)) + C = F(x) + C$$

Formales Vorgehen:

(i) Gesucht: $\int f(x)\,dx = \int \phi(g(x))\frac{dg}{dx}(x)\,dx$

(ii) Substitution: $u = g(x)$

$$\longrightarrow \frac{du}{dx} = \frac{dg}{dx} \longrightarrow dx = \frac{1}{\frac{dg}{dx}}\,du$$

(iii) Einsetzen:

$$\int \phi(g(x)) \frac{dg}{dx}(x)\,dx = \int \phi(u) \frac{dg}{dx}(x) \frac{1}{\frac{dg}{dx}(x)}\,du$$

$$= \int \phi(u)\,du = \Phi(u) + C$$

(iv) Rücksubstitution:

$$\underset{(u=g(x))}{=} \Phi(g(x)) + C$$

$$= F(x) + C$$

$$\left(\text{I.e.} \quad \int f(x)\,dx = F(x) + C \right)$$

Bsp: $\int \frac{x}{\sqrt{1-x^2}}\,dx$ Subs. versuche: $u = \sqrt{1-x^2}$

$$\longrightarrow \frac{du}{dx} = \frac{1}{2\sqrt{1-x^2}}(-2x)$$

$$\longrightarrow dx = -\frac{\sqrt{1-x^2}}{x}\,du$$

Einsetzen:

$$\underline{\int \frac{x}{\sqrt{1-x^2}}\,dx} = \int \frac{x}{u}\left(-\frac{\sqrt{1-x^2}}{x}\right)du$$

$$= -\int \frac{\sqrt{1-x^2}}{u}\,du$$

$$= -\int \frac{u}{u}\,du = -\int du = -u + C$$

$$\underline{= -\sqrt{1-x^2} + C}$$

$$\left(\text{Rücksubst.: } u = \sqrt{1-x^2}\right)$$

Gleiches Bsp., andere Variante:

$\int \frac{x}{\sqrt{1-x^2}}\,dx$ Subs.: $x = \sin(u)$

$$\longrightarrow \frac{dx}{du} = \cos(u) \longrightarrow dx = \cos(u)\,du$$

$$\longrightarrow \underline{\int \frac{x}{\sqrt{1-x^2}}\,dx} = \int \frac{\sin(u)}{\sqrt{1-\sin^2(u)}} \cos(u)\,du$$

$$= \int \frac{\sin(u)}{\cancel{\cos(u)}} \cancel{\cos(u)}\,du = \int \sin(u)\,du$$

$$= -\cos(u) + C$$

$$= -\sqrt{1-\sin^2(u)} + C$$

$$\underline{= -\sqrt{1-x^2} + C}$$

Aufgabe: $\int \frac{6x^2}{(1-4x^3)^3}\,dx$ mit Subtitution: $u = 1-4x^3$

Lösg: $u = 1-4x^3 \longrightarrow \frac{du}{dx} = -12x^2 \longrightarrow dx = -\frac{du}{12x^2}$

$$\longrightarrow \underline{\int \frac{6x^2}{(1-4x^2)^3}\,dx} = -\int \frac{6x^2}{u^3}\,\frac{du}{12x^2}$$

$$= -\frac{1}{2}\int \frac{1}{u^3}\,du = -\frac{1}{2}\cdot\frac{1}{(-2)}\,u^{-2} + C$$

$$= \frac{1}{4}\,u^{-2} + C = \underline{\frac{1}{4}\cdot\frac{1}{(1-4x^3)^2} + C}$$

Alternative: $u = (1-4x^3)^3$

$$\longrightarrow \frac{du}{dx} = 3(1-4x^3)^2(-12x^2) \longrightarrow dx = \frac{du}{3(1-4x^3)^2(-12x^2)}$$

$$\longrightarrow \int \frac{6x^2}{(1-4x^2)^3}\,dx = \int \frac{6x^2}{u}\,\frac{du}{3\underline{(1-4x^3)^2}(-12x^2)}$$

$$= \int -\frac{1}{2}\,\frac{du}{u\,3\,u^{\frac{2}{3}}}$$

$$= -\frac{1}{6}\int u^{-\frac{5}{3}}\,du = -\frac{1}{6}\,u^{-\frac{2}{3}}\cdot\left(-\frac{3}{2}\right) + C$$

$$= \frac{3}{12}\left((1-4x^3)^3\right)^{-\frac{2}{3}} + C$$

$$= \underline{\frac{1}{4}\cdot\frac{1}{(1-4x^3)^2} + C}$$

Aufgabe: Man bestimme $\int \sin^2(x)\cos(x)\,dx$ mit Subst.

Lösg: $u = \sin(x) \longrightarrow \frac{du}{dx} = \cos(x) \longrightarrow dx = \frac{du}{\cos(x)}$

$$\longrightarrow \underline{\int \sin^2(x)\cos(x)\,dx} = \int u^2 \cancel{\cos(x)}\,\frac{du}{\cancel{\cos(x)}} = \int u^2\,du$$

$$= \frac{u^3}{3} + C$$

$$= \underline{\frac{\sin^3(x)}{3} + C}$$

Aufgabe: Man bestimme $\int \frac{6x^2-5}{2x^3-5x+6}\,dx$ mit Subst.

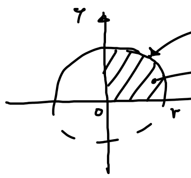
Lösg: $u = 2x^3-5x+6 \longrightarrow \frac{du}{dx} = 6x^2-5 \longrightarrow dx = \frac{du}{6x^2-5}$

$$\longrightarrow \underline{\int \frac{6x^2-5}{2x^3-5x+6}\,dx} = \int \frac{6x^2-5}{u}\cdot\frac{du}{6x^2-5} = \int \frac{du}{u}$$

$$= \log(u) + C$$

$$= \underline{\log(2x^3-5x+6) + C}$$

Beispiel zu Subst. mit Grenzen (bestimmter Integral):



$y = f(x) = \sqrt{r^2 - x^2}$

$A = ?$ $\longrightarrow A = \int_0^r \sqrt{r^2 - x^2}\, dx$

Subst.: $x = r\sin(u)$

$\longrightarrow \frac{dx}{du} = r\cos(u) \longrightarrow dx = r\cos(u)\,du$

Grenzen substituieren:

$x_1 = 0 \longrightarrow$ wieviel ist dann u?

$0 = r\sin(u) \longrightarrow \underline{u_1} = \arcsin(0) = \underline{0}$

$x_2 = r \longrightarrow$ wieviel ist dann u?

$r = r\sin(u) \longrightarrow \underline{u_2} = \arcsin(1) = \underline{\frac{\pi}{2}}$

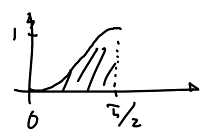
$$\longrightarrow A = \int_0^r \sqrt{r^2 - x^2}\, dx$$

$$= \int_0^{\pi/2} \sqrt{\underbrace{r^2 - r^2\sin^2(u)}_{r^2(1-\sin^2(u))}}\, r\cos(u)\,du = \int_0^{\pi/2} r\sqrt{\underbrace{1 - \sin^2(u)}_{=\cos^2(u)}}\, r\cos(u)\,du$$

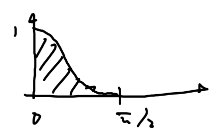
$$= \int_0^{\pi/2} r^2 \cos^2(u)\,du$$

<u>Trick</u>:

$$\int_0^{\pi/2} \sin^2(u)\,du = ?$$



$$\int_0^{\pi/2} \cos^2(u)\,du = ?$$



Beide Flächen sind gleich gross!

Mit: $\int_0^{\pi/2} \sin^2(x)\,dx + \int_0^{\pi/2} \cos^2(x)\,dx = \int_0^{\pi/2} \underbrace{\left(\sin^2(x) + \cos^2(x)\right)}_{=1} dx$

$$= \int_0^{\pi/2} dx = \frac{\pi}{2}$$

$$\longrightarrow \int_0^{\pi/2} \cos^2(x)\,dx = \frac{\pi}{4}$$

$$\longrightarrow \underline{\underline{A}} = r^2 \int_0^{\pi/2} \cos^2(u)\,du = \underline{\underline{r^2 \frac{\pi}{4}}}$$