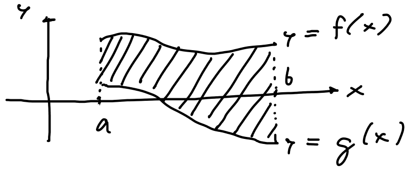
Wiederholung

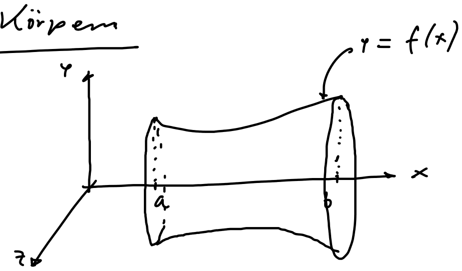
Anwendungen der Integralrechnung

Flächenberechnung



$$A = \int_a^b \underbrace{(f(x) - g(x))\,dx}_{dA}$$

Volumen von Rot.-Körpern



$$V = \int_a^b \overbrace{\underbrace{f(x)^2\,\pi}_{A(x)}\,dx}^{dV}$$

Volumenformel hat Form:

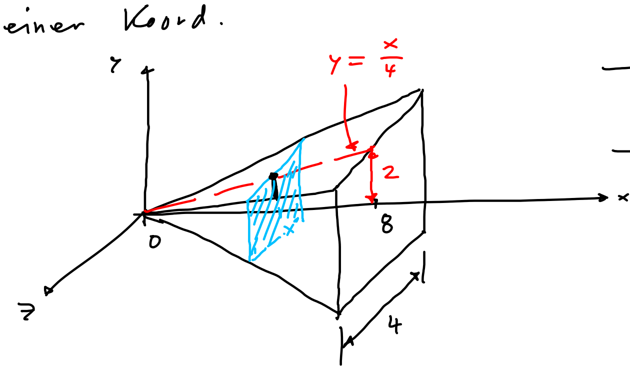
$$V = \int_a^b \underbrace{\pi f(x)^2\,dx}_{dV}$$

$$dV = \underbrace{\pi f(x)^2}\,dx = A(x)\,dx$$

Querschnittsfläche: $A(x)$

→ Idee funktioniert auch für Körper mit Querschnittsfläche in Abh. einer Koord.

Bsp.:



→ $A(x) = \left(2\,\frac{x}{4}\right)^2 = \frac{x^2}{4}$

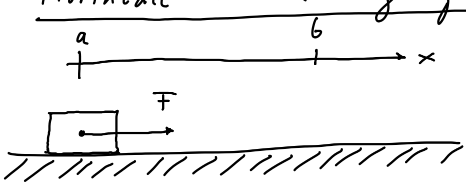
→ $V = \int_0^8 A(x)\,dx$

$$= \int_0^8 \frac{x^2}{4}\,dx = \left.\frac{x^3}{12}\right|_0^8 = \frac{8\cdot 64}{12}$$

$$= \frac{128}{3}$$

Mechanische Arbeit

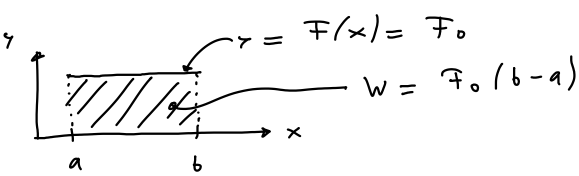
Bsp: Horizontale Beschleunigung einer Masse:



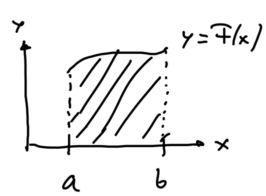
Annahme: konstante Kraft F_0

W = Arbeit = Kraft · Weg

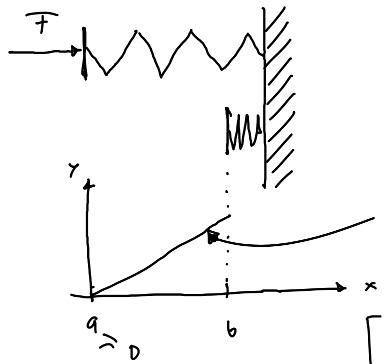
↓ in Wegrichtung



Allgemein: $\boxed{W = \int_a^b F(x)\,dx}$

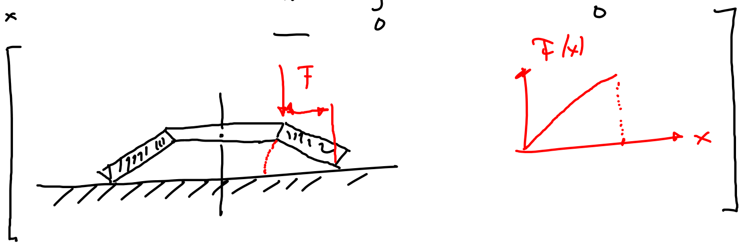


Bsp: Gespeicherte Energie in Feder (Federarbeit)



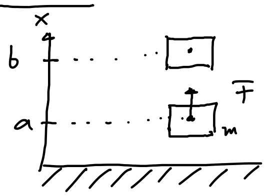
Federkraft.

$y = kx = F(x) \longrightarrow \underline{W} = \int_0^b F(x)\,dx = \int_0^b kx\,dx = k\frac{x^2}{2}\Big|_0^b = \underline{\frac{kb^2}{2}}$

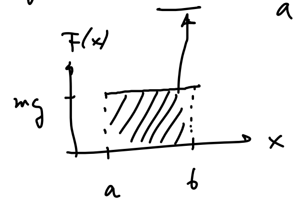


Gravitationsarbeit

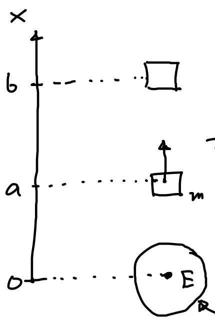
(i) Erdnähe:



$F = mg \longrightarrow \underline{W} = \int_a^b mg\,dx = \underline{mg(b-a)}$



(ii) Allgemein:

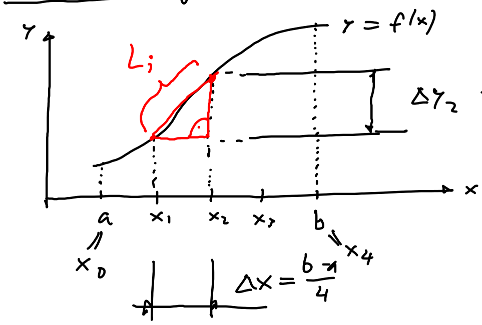


Erde mit Masse M_E

$F = \frac{mM_E}{x^2} \longrightarrow \underline{W} = \int_a^b F(x)\,dx \overset{\text{Aufgabe}}{=} \int_a^b \frac{mM_E}{x^2}\,dx$

$= mM_E\left(-\frac{1}{x}\right)\Big|_a^b = \underline{mM_E\left(\frac{1}{a} - \frac{1}{b}\right)}$

Bogenlänge eines Graphen



$\Delta x = \frac{b-a}{4}$

Idee: Approx. mit Geradenstücken

$\Delta y_2 = f(x_2) - f(x_1)$

$\longrightarrow L_i = \sqrt{\Delta x^2 + \Delta y_i^2}$

$= \Delta x\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}$

$\longrightarrow L_n = \sum_{i=1}^n L_i = \sum_{i=1}^n \Delta x\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}$

$= \frac{b-a}{n}\sum_{i=1}^n \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}$

$\xrightarrow{n\to\infty} \frac{dy}{dx} = \frac{df}{dx}$

$\longrightarrow \boxed{L = \lim_{n\to\infty} L_n = \int_a^b \sqrt{1 + \left(\frac{df}{dx}(x)\right)^2}\,dx}$

Aufgabe: Länge des Graphen von $y = \frac{2}{3}(x-1)^{3/2}$

für $1 \le x \le 4$.

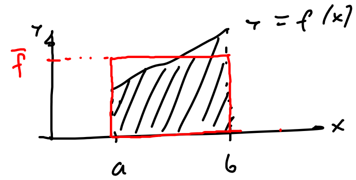
Lösg.: $f(x) = \frac{2}{3}(x-1)^{3/2} \longrightarrow \frac{df}{dx}(x) = (x-1)^{\frac{1}{2}}$

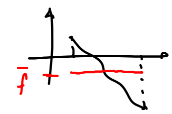
$$\longrightarrow \underline{L} = \int_1^4 \sqrt{1 + \left((x-1)^{\frac{1}{2}}\right)^2}\,dx = \int_1^4 \sqrt{x}\,dx = \frac{2}{3}\,x^{\frac{3}{2}}\Big|_1^4$$

$$= \frac{2}{3}(8-1) = \underline{\frac{14}{3}}$$

Durchschnitt einer Fkt. f auf Intervall $[a,b]$ ist:

$$\boxed{\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$





Aufgabe: Ges: Durchschnitt von $f(x) = \sin(x)$

auf $\left[0, \frac{3\pi}{2}\right]$.

Lösg: $$\underline{\bar{f}} = \frac{1}{\frac{3\pi}{2}}\int_0^{\frac{3\pi}{2}} \sin(x)\,dx = \frac{2}{3\pi}\left(-\cos(x)\right)\Big|_0^{\frac{3\pi}{2}} = \underline{\frac{2}{3\pi}}$$