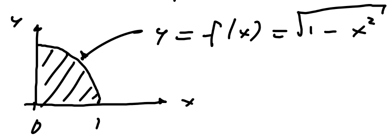
Bsp: Flächeninterpretation kann Berechnung stark vereinfachen:

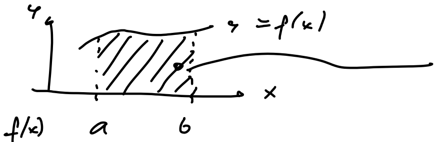


$\longrightarrow \int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}.$

$$\left[\begin{array}{l} y = \sqrt{1-x^2} \\ y^2 = 1-x^2 \\ x^2+y^2 = 1 \end{array}\right]$$

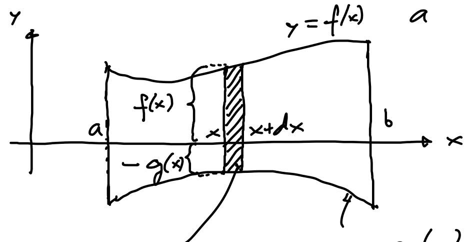
Anwendungen der Integralrechnung

Flächeninhalt



$\int_a^b f(x)\,dx$

Bsp:

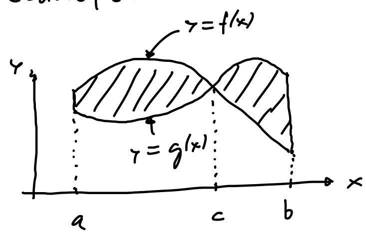


$dA = (f(x) - g(x))\,dx$

⚠ gilt falls $f(x) \geq g(x)$.

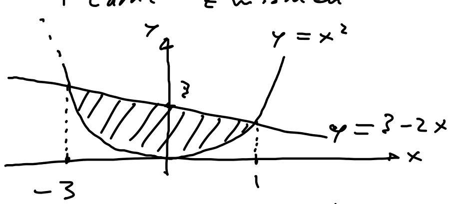
Gesamtfläche: Aufaddieren von dA: $A = \int_a^b (f(x) - g(x))\,dx$

Bsp:



$$A = \underbrace{\int_a^c (f(x) - g(x))\,dx}_{f(x) \geq g(x)} + \underbrace{\int_c^b (g(x) - f(x))\,dx}_{g(x) \geq f(x)}$$

Bsp: Fläche zwischen Graphen von $f(x) = x^2$ & $g(x) = 3 - 2x$.



Schnittpkt.: $x^2 \stackrel{!}{=} 3 - 2x$

$x^2 + 2x - 3 = 0$

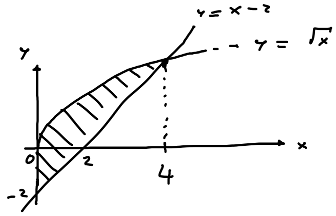
$\longrightarrow x_\pm = \frac{-2 \pm \sqrt{4+12}}{2} = 1, -3$

$$\longrightarrow \underline{\underline{A}} = \int_{-3}^{1} (3 - 2x - x^2)\,dx = \left(3x - x^2 - \frac{x^3}{3}\right)\Big|_{-3}^{1}$$

$$= \ldots = \underline{\underline{\frac{32}{3}}}$$

Aufgabe: Ges: Fläche begrenzt durch: $x = 0$
$y = \sqrt{x}$
$y = x - 2$

Lösg.:



Schnittpkt.: $\sqrt{x} \stackrel{!}{=} x - 2 \quad (*)$

$\longrightarrow x = x^2 - 4x + 4$

$\longrightarrow x^2 - 5x + 4 = 0$

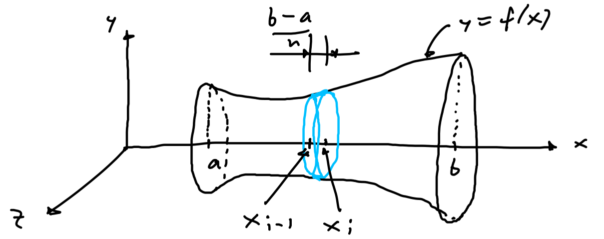
$\longrightarrow x_\pm = \frac{5 \pm \sqrt{25-16}}{2} = \frac{5 \pm 3}{2}$

$\longrightarrow \frac{x_+ = 4}{x_- = 1}$ keine Lösg. von $(*)$

$$A = \int_0^4 (\sqrt{x} - x + 2)\,dx = \int_0^4 \left(x^{\frac{1}{2}} - x + 2\right)dx$$

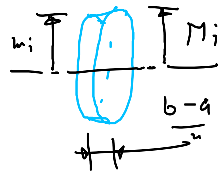
$$= \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{2} + 2x\right)\Big|_0^4 = \left(\frac{2}{3}\cdot 8 - 8 + 8\right) - 0 = \underline{\frac{16}{3}}$$

Volumen von Rotationskörpern



Idee: In n Scheiben schneiden, senkrecht zur Rot.-Achse.

i-te Scheibe:



$$m_i^2 \pi \frac{b-a}{n} \le V_S \le M_i^2 \pi \frac{b-a}{n}$$

$\longrightarrow$

$$\frac{b-a}{n} \sum_{i=1}^{n} m_i^2 \pi \le V \le \frac{b-a}{n} \sum_{i=1}^{n} M_i^2 \pi$$

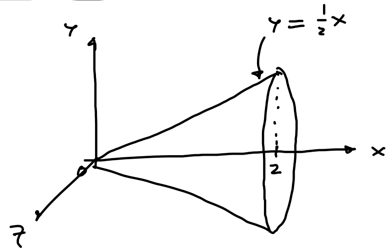
$\downarrow n \to \infty$ $\qquad\qquad$ $\downarrow n \to \infty$

$$\int_a^b f(x)^2 \pi \, dx \qquad\qquad \int_a^b f(x)^2 \pi \, dx$$

$\longrightarrow$ $$\boxed{V = \underline{\pi} \int_a^b \underline{f(x)^2 \, dx}}$$

$\longrightarrow$ $\underline{f(x)^2 \pi \, dx = dV}$: infinitesimales Volumen

Bsp: Kegel

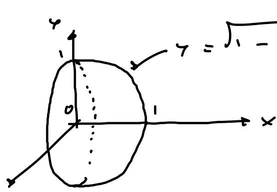


$$V = \pi \int_0^2 f(x)^2 \, dx$$
$$= \pi \int_0^2 \left(\tfrac{1}{2}x\right)^2 dx$$
$$= \frac{\pi}{4} \int_0^2 x^2 \, dx = \frac{\pi}{4} \cdot \frac{x^3}{3} \Big|_0^2$$
$$= \frac{\pi}{4}\left(\frac{8}{3} - 0\right) = \underline{\underline{\frac{2\pi}{3}}}$$

Aufgabe: Ges: (i) Volumen Kugel mit $r=1$ (Hinweis: Zuerst Halbkugel).

(ii) —— " —— allg. Radius r.

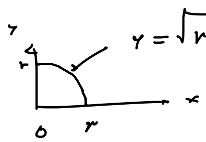
Lösg: (i) Halbkugel:



$$V_H = \pi \int_0^1 f(x)^2 \, dx$$
$$= \pi \int_0^1 (1-x^2) \, dx$$
$$= \pi \left(x - \frac{x^3}{3}\right)\Big|_0^1 = \pi\left(1 - \frac{1}{3}\right) = \underline{\frac{2\pi}{3}}$$

$$\longrightarrow \underline{V_{Kugel} = \frac{4}{3}\pi}$$

(ii)



$$V_H = \pi \int_0^r f(x)^2 \, dx = \pi \int_0^r (r^2 - x^2) \, dx = \pi \left(r^2 x - \frac{x^3}{3}\right)\Big|_0^r$$
$$= \pi\left(r^3 - \frac{r^3}{3}\right) = \frac{2\pi r^3}{3}$$

$$\longrightarrow \underline{V_{Kugel} = \frac{4}{3}\pi r^3}$$