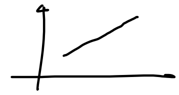
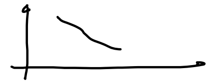
Wiederholung:

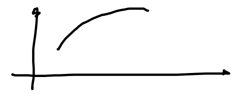
Bedeutung der Ableitungen

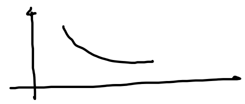
1. Ableitung

$\frac{df}{dx}(x) > 0$  streng monoton steigend

$\frac{df}{dx}(x) < 0$  —"— fallend

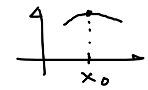
2. Ableitung

$\frac{d^2f}{dx^2}(x) < 0$  Rechtskurve

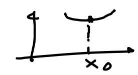
$\frac{d^2f}{dx^2}(x) > 0$  Linkskurve

Extrema:

$$\frac{df}{dx}(x_0) = 0 \quad \& \quad \frac{d^2f}{dx^2}(x_0) < 0 \Rightarrow (x_0, f(x_0)) \text{ Max.}$$



$$\frac{df}{dx}(x_0) = 0 \quad \& \quad \frac{d^2f}{dx^2}(x_0) > 0 \Rightarrow (x_0, f(x_0)) \text{ Min.}$$



Aufgabe: $f(x) = \frac{x^2}{1+x^2}$ Ges: Min./Max.

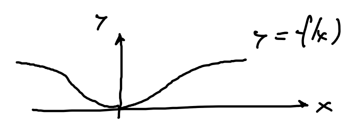
Lösg.:

$$\frac{df}{dx}(x) = \frac{2x}{1+x^2} - \frac{x^2 2x}{(1+x^2)^2} = \frac{2x(1+x^2) - 2x\cdot x^2}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$$

$$\frac{d^2f}{dx^2}(x) = \frac{2}{(1+x^2)^2} - \frac{2x\,2(1+x^2)2x}{(1+x^2)^4} = \frac{2(1+x^2) - 8x^2}{(1+x^2)^3} = \frac{2-6x^2}{(1+x^2)^3}$$

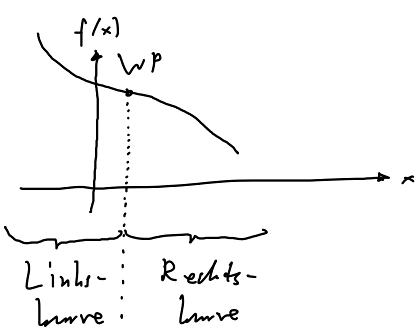
$$\frac{df}{dx}(x) = \frac{2x}{(1+x^2)^2} \stackrel{!}{=} 0 \longrightarrow \underline{x_0 = 0}$$

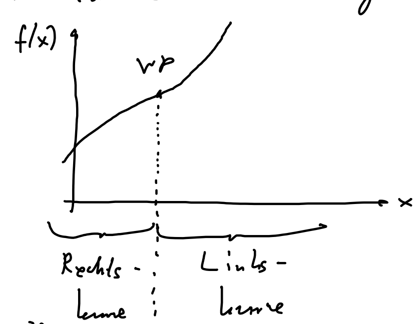
$$\frac{d^2f}{dx^2}(x_0) = 2 > 0 \longrightarrow \underline{\text{bei } x_0 \text{ ist lok. Min.}}$$

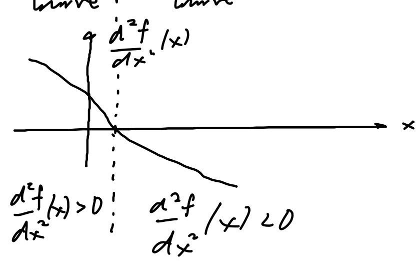


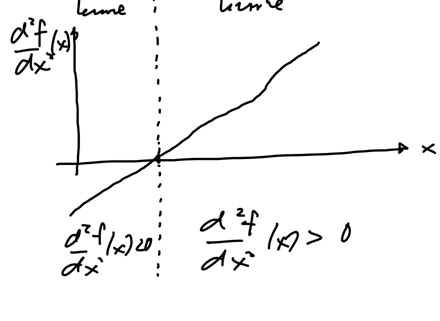
$$y_0 = 0 \longrightarrow (x_0, f(x_0)) = \underline{(0, 0)}$$

Wendepunkte (WP): Idee: Graph geht von Links- in Rechtskurve oder umgekehrt.

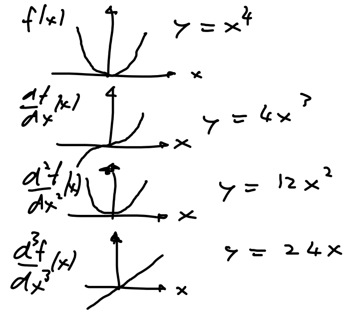








Wir sehen: x_0 WP $\Rightarrow \frac{d^2f}{dx^2}(x_0) = 0$.



Aber: $\cancel{\Leftarrow}$ Bsp.: $f(x) = x^4$

$$\downarrow$$

$$\frac{d^2f}{dx^2}(0) = 0$$

aber $x_0 = 0$ ist kein WP!

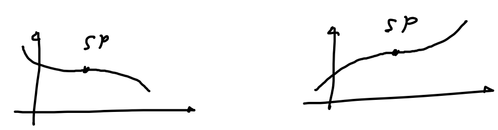
Wie findet man nun einen WP?

Wir haben:

$$\boxed{\frac{d^2f}{dx^2}(x_0) = 0 \quad \& \quad \frac{d^3f}{dx^3}(x_0) \neq 0 \quad \Rightarrow (x_0, f(x_0)) \text{ WP}}$$

Sattelpkt. (SP)

sind WP mit horizontalen Tang.



$$\boxed{\frac{df}{dx}(x_0) = 0 \quad \& \quad \frac{d^2f}{dx^2}(x_0) = 0 \quad \& \quad \frac{d^3f}{dx^3}(x_0) \neq 0 \quad \Rightarrow (x_0, f(x_0)) \text{ SP}}$$

Aufgabe: Extremalstellen & WP von $f(x) = -\frac{2}{3}x^3 + 2x^2 - 2x + 2$

Lösg.: $\frac{df}{dx}(x) = -2x^2 + 4x - 2 \overset{!}{=} 0 \rightarrow \underline{x_1} = \frac{-4 \pm \sqrt{16-16}}{-4} = \underline{1}$

$\underline{y_1} = f(x_1) = -\frac{2}{3} + 2 - 2 + 2 = \underline{\frac{4}{3}}$

$\frac{d^2f}{dx^2}(x) = -4x + 4 \longrightarrow \frac{d^2f}{dx^2}(x_1) = -4 + 4 = 0$

$\frac{d^3f}{dx^3}(x) = -4 \neq 0 \longrightarrow \underline{(1, \frac{4}{3}) \text{ ist SP}}$ (dies ist auch $\underline{\text{WP}}$)

Aufgabe: Man charakterisiere alle Pkte. mit horizontaler Tang. von $f(x) = x^4 - 4x^3$

Lösg.: $\frac{df}{dx}(x) = 4x^3 - 12x^2 \overset{!}{=} 0$

$4x^2(x-3) = 0 \longrightarrow \underline{x_1 = 3}$, $\underline{y_1} = f(x_1) = \underline{-27}$

$\underline{x_2 = 0}$, $\underline{y_2} = f(x_2) = \underline{0}$

$\frac{d^2f}{dx^2}(x) = 12x^2 - 24x$

$\longrightarrow \frac{d^2f}{dx^2}(x_1) = 108 - 72 = \ldots > 0 \longrightarrow \underline{(3, -27) \text{ ist Min.}}$

$\frac{d^2f}{dx^2}(x_2) = 0$

$\frac{d^3f}{dx^3}(x) = 24x - 24 \longrightarrow \frac{d^3f}{dx^3}(x_2) = -24 \longrightarrow \underline{(0,0) \text{ ist SP}}$

Bemerkung: Allgemeines Kriterium für Aufsuchen von rel. Max., Min., SP, WP:

(i) Finde x_0 mit $\frac{df}{dx}(x_0) = 0$

(ii) Berechne: $\frac{d^2f}{dx^2}(x_0)$, $\frac{d^3f}{dx^3}(x_0), \ldots, \frac{d^nf}{dx^n}(x_0)$,

bis $\frac{d^nf}{dx^n}(x_0) \neq 0$.

$\hookrightarrow$ n gerade, $\frac{d^nf}{dx^n}(x_0) < 0 \Rightarrow$ Max.

$\frac{d^nf}{dx^n}(x_0) > 0 \Rightarrow$ Min.

$\hookrightarrow$ n ungerade $\Rightarrow$ SP

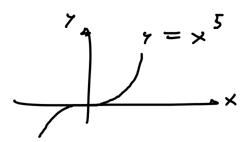
(iii) WP suchen: Suche x_1 mit $\frac{d^2f}{dx^2}(x_1) = 0$

(iv) (a) $\frac{d^3f}{dx^3}(x_1) \neq 0 \Rightarrow$ WP

(b) $\frac{d^2f}{dx^2}(x_1 - \varepsilon)$, $\frac{d^2f}{dx^2}(x_1 + \varepsilon)$ haben unterschiedliche Vorzeichen $\Rightarrow$ WP

Bsp.: (i) $f(x) = x^5$

(ii) $f(x) = x^6$



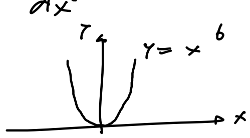
(i) $\frac{df}{dx}(x) \stackrel{!}{=} 0 \longrightarrow x = 0$

$\frac{d^2f}{dx^2}(0) = \ldots = \frac{d^4f}{dx^4}(0) = 0$ aber $\frac{d^5f}{dx^5}(0) = 120$

5 ist ungerade $\Rightarrow$ SP bei $x = 0$

(ii) $\frac{df}{dx}(x) \stackrel{!}{=} 0 \rightarrow x = 0$

$\frac{d^2f}{dx^2}(0) = \ldots = \frac{d^5f}{dx^5}(0) = 0$ aber $\frac{d^6f}{dx^6}(0) = 720 > 0$

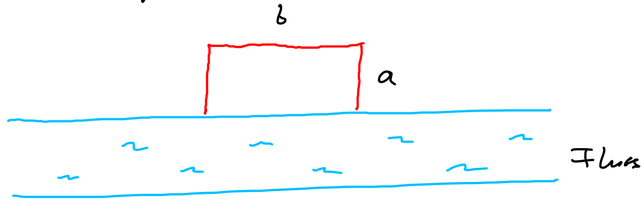


und 6 ist gerade $\Rightarrow$ lok. Min. bei $x = 0$

Extremalwertprobleme (Optimierungsaufgaben)

Bsp.: Mit 240 m Draht größtmögliches rechteckiges Feld am Fluss einzäunen:

Ges: a, b



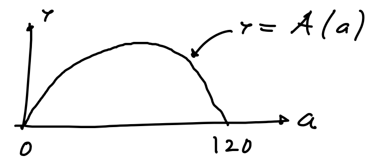
Was wird maximiert: $A = a \cdot b$ ← Zielfunktion

Nebenbedingung: $2a + b = 240$

$\longrightarrow b = 240 - 2a$

$\longrightarrow A(a) = a \cdot (240 - 2a)$

$= 240a - 2a^2$



$\frac{dA}{da}(a) = 240 - 4a \overset{!}{=} 0 \longrightarrow \underline{a = 60} \longrightarrow \underline{b} = 240 - 2 \cdot 60 = \underline{120}$

Aufgabe: Herstellung einer zylindrischen Büchse (mit Deckel), so dass Materialverbrauch minimal ist.

Volumen der Büchse $= 1\,dm^3$

Ges: Verhältnis von Durchmesser zu Höhe.