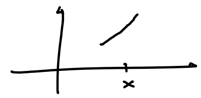
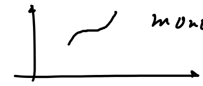
Bedeutung der Ableitungen



streng monoton



monoton

1. Ableitung

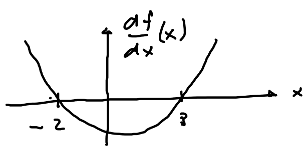
$$\frac{df}{dx}(x) > 0 \Rightarrow \text{Funktion an der Stelle } x \text{ streng monoton steigend}$$

$$\frac{df}{dx}(x) < 0 \Rightarrow \text{———— " ———— streng monoton fallend}$$

Aufgabe: Wo ist $f(x) = 2x^3 - 3x^2 - 36x + 5$ streng monoton steigend?

Lösg.: $\frac{df}{dx}(x) = 6x^2 - 6x - 36 > 0$

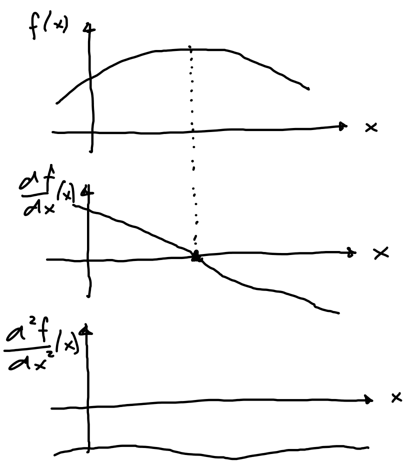
$$\longleftrightarrow \underbrace{x^2 - x - 6}_{\overset{!}{=}0 \,\longrightarrow\, x \,=\, \frac{1 \pm \sqrt{1+24}}{2} \,=\, \underline{3, -2}} > 0$$



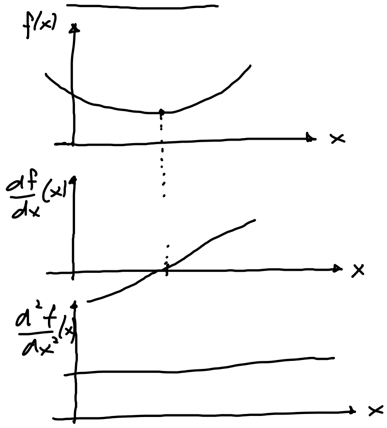
$\longrightarrow$ Für $x \in \mathbb{R} \setminus [-2, 3]$ ist $f(x)$ streng mon. steigend.

2. Ableitung

Rechtskurve (konkav)



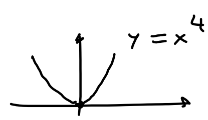
Linkskurve (konvex)



$$\frac{d^2f}{dx^2}(x) < 0 \Rightarrow \text{Rechtskurve}$$

$$\frac{d^2f}{dx^2}(x) > 0 \Rightarrow \text{Linkskurve}$$

Bem:



$f(x) = x^4$

$\frac{df}{dx}(x) = 4x^3$

$\frac{d^2f}{dx^2}(x) = 12x^2 \rightarrow \frac{d^2f}{dx^2}(0) = 0$

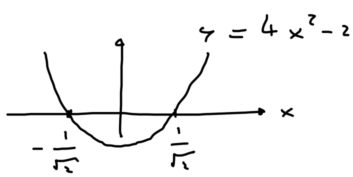
Aufgabe:
Wo ist der Graph von $f(x) = e^{-x^2}$ eine Rechtskurve?

Lösg.: $\frac{df}{dx}(x) = -2x\, e^{-x^2}$

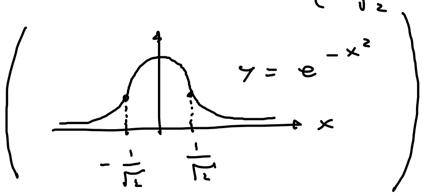
$$\frac{d^2f}{dx^2}(x) = -2e^{-x^2} + 4x^2 e^{-x^2} = e^{-x^2}(4x^2 - 2) < 0$$

für Rechtskurve

$$\longleftrightarrow \underbrace{4x^2 - 2}_{\overset{!}{=}0 \,\longrightarrow\, x \,=\, \pm \frac{1}{\sqrt{2}}} < 0$$



$\longrightarrow$ Für $x \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ ist $y = e^{-x^2}$ Rechtskurve.



Aufgabe: Wo ist der Graph von $f(x) = 2x^3 - 3x^2 - 36x + 5$ eine Rechtskurve?

Lösg.:

$$\frac{df}{dx}(x) = 6x^2 - 6x - 36$$

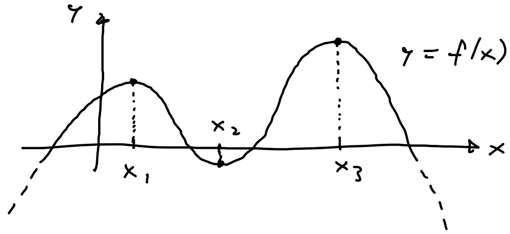
$$\frac{d^2f}{dx^2}(x) = 12x - 6 < 0 \longrightarrow 12x < 6$$

$$x < \frac{1}{2}$$

$\longrightarrow$ Für $x < \frac{1}{2}$ ist $y = f(x)$ Rechtskurve.

Relative & absolute Extremalstellen

Bsp:

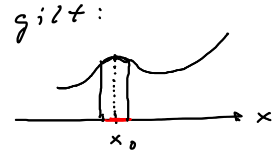


relative Max.: x_1, x_3
— " — Min.: x_2
absolute Max.: x_3
— " — Min.: ex. nicht

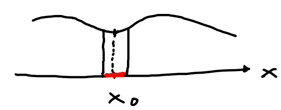
} Extremalstellen

Def: Falls für alle x in einer Umgebung von x_0 ($x \neq x_0$) gilt:

(i) $f(x) < f(x_0) \Rightarrow (x_0, f(x_0))$ ist rel. Max.

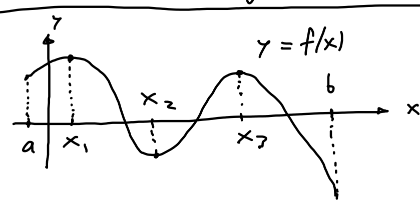


(ii) $f(x) > f(x_0) \Rightarrow (x_0, f(x_0))$ ist rel. Min.



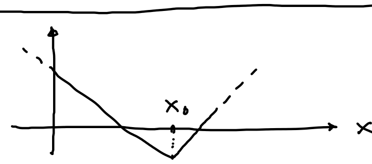
Bem. (spezielle Fälle)

(i) Fkt. auf abgeschlossenem Intervall $[a, b]$:



rel. Max.: x_1, x_3
rel. Min.: a, b, x_2
abs. Max.: x_1
abs. Min.: b

(ii) nicht diff'bare Fkt.:

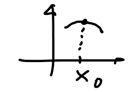


rel./abs. Min. bei x_0

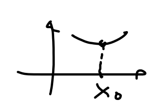
Hinreichende Bedingung für Max./Min.

Betrachten: Diff'bare Fkt. an Stelle x_0, x_0 kein Randpkt.

(i) $\frac{df}{dx}(x_0) = 0$ & $\frac{d^2f}{dx^2}(x_0) < 0 \Rightarrow (x_0, f(x_0))$ rel. Max.



(ii) $\frac{df}{dx}(x_0) = 0$ & $\frac{d^2f}{dx^2}(x_0) > 0 \Rightarrow (x_0, f(x_0))$ rel. Min.



Aufgabe: $f(x) = 2x^3 - 3x^2 - 36x + 5$

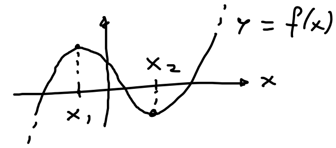
Ges: lok. Max./Min.

Lösg.: $\frac{df}{dx}(x) = 6x^2 - 6x - 36 \overset{!}{=} 0 \longrightarrow \underline{x_1 = -2}\,;\ \underline{x_2 = 3}$

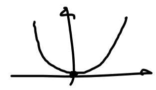
$\frac{d^2f}{dx^2}(x) = 12x - 6 \longrightarrow \frac{d^2f}{dx^2}(x_1) = -24 - 6 = -30 \longrightarrow$ lok. Max. bei x_1

$\frac{d^2f}{dx^2}(x_2) = 36 - 6 = 30 \longrightarrow$ lok. Min. bei x_2

y-Werte: $f(x_1) = \dots = \underline{49}\,;\ f(x_2) = \dots = \underline{-76}$



Bem.: $\not\Leftarrow$: Bsp: $f(x) = x^4$



$\longrightarrow$ Min. bei $x_0 = 0$.

aber: $\underline{\frac{d^2f}{dx^2}(0) = 0}$

Aufgabe: $f(x) = \frac{x^2}{1 + x^2}$ Ges: Min./Max.