Wiederholung:

Konstante: $\frac{d}{dx} C = 0$

Potenz: $\frac{d}{dx} x^n = n x^{n-1}$, $n \in \mathbb{Z} \setminus \{0\}$

Summe: $\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{df}{dx}(x) + \frac{dg}{dx}(x)$

Konst. Fakt.: $\frac{d}{dx}\left(C f(x)\right) = C \frac{df}{dx}(x)$

Wurzel: $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

Trigo: $\frac{d}{dx}\sin(x) = \cos(x)$

$\frac{d}{dx}\cos(x) = -\sin(x)$

Exp.: $\frac{d}{dx} e^x = e^x$

Produkt:

$$\frac{d}{dx}\left(f(x)\, g(x)\right) = \frac{df}{dx}(x)\, g(x) + f(x)\, \frac{dg}{dx}(x)$$

Quotient:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{df}{dx}(x)\, g(x) - f(x)\, \frac{dg}{dx}(x)}{g(x)^2}$$

Kettenregel: $F(x) = f(g(x))$

$$\frac{dF}{dx}(x) = \frac{d}{dx} f(g(x)) = \underbrace{\frac{df}{dx}(g(x))}_{\text{Äussere Ableitung}}\ \underbrace{\frac{dg}{dx}(x)}_{\text{Innere Ableitung}}$$

Bsp: (i) $F(x) = (2x+1)^3$ ist zusammengesetzt aus:

$$\left.\begin{aligned} f(x) &= x^3 \\ g(x) &= 2x+1 \end{aligned}\right\} F(x) = f(g(x))$$

$f(x) = x^3$

$\longrightarrow \frac{df}{dx}(x) = 3x^2$

$\longrightarrow \frac{df}{dx}(g(x)) = 3(g(x))^2$

$\longrightarrow \frac{df}{dx}(2x+1) = 3(2x+1)^2$

$$\frac{dF}{dx}(x) = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x)$$

$$= 3(2x+1)^2 \cdot 2 = 6(2x+1)^2$$

(ii) $F(x) = \sin^2(x)$ ist zusammengesetzt aus:

$$\left.\begin{aligned} f(x) &= x^2 \\ g(x) &= \sin(x) \end{aligned}\right\} F(x) = f(g(x))$$

$f(x) = x^2$

$\longrightarrow \frac{df}{dx}(x) = 2x$

$\longrightarrow \frac{df}{dx}(g(x)) = 2\, g(x)$

$\longrightarrow \frac{df}{dx}(\sin(x)) = 2\sin(x)$

$$\frac{dF}{dx}(x) = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x)$$

$$= 2\sin(x) \cdot \cos(x)$$

(iii) $F(x) = \sqrt{1+x^3}$

$$\left.\begin{aligned} f(x) &= \sqrt{x} \\ g(x) &= 1+x^3 \end{aligned}\right\} F(x) = f(g(x))$$

$$\frac{dF}{dx}(x) = \frac{df}{dx}(g(x))\, \frac{dg}{dx}(x)$$

$$= \frac{1}{2\sqrt{1+x^3}} \cdot 3x^2 = \frac{3x^2}{2\sqrt{1+x^3}}$$

Kettenregel

Aufgabe: Ges: Ableitung von: (i) $F(x) = e^{\sin(x)}$

(ii) $G(x) = \frac{1}{\cos(x)}$

Lösg.: (i) $f(x) = e^x$, $g(x) = \sin(x)$ $\Big\}$ $F(x) = f(g(x))$; $\frac{dF}{dx}(x) = \frac{df}{dx}(g(x))\,\frac{dg}{dx}(x)$

$f(x) = e^x \longrightarrow \frac{df}{dx}(x) = e^x \longrightarrow \frac{df}{dx}(\sin(x)) = e^{\sin(x)}$ $\quad = e^{\sin(x)} \cos(x)$

(ii) $f(x) = \frac{1}{x}$, $g(x) = \cos(x)$ $\Big\}$ $G(x) = f(g(x))$; $\frac{dG}{dx}(x) = \frac{df}{dx}(g(x))\,\frac{dg}{dx}(x)$

$f(x) = \frac{1}{x} = x^{-1}$ $\quad \frac{d}{dx}x^n = n x^{n-1}$

$\longrightarrow \frac{df}{dx}(x) = -1 \cdot x^{-2} = -\frac{1}{x^2}$

$\longrightarrow \frac{df}{dx}(\cos(x)) = -\frac{1}{\cos^2(x)}$

$$= -\frac{1}{\cos^2(x)}(-\sin(x))$$

$$= \frac{\sin(x)}{\cos^2(x)}$$

Logarithmus: Betrachten:

$$1 = \frac{d}{dx} x = \frac{d}{dx}\Big(\underbrace{e^{\log(x)}}_{=x}\Big) = \overbrace{e^{\log(x)}}^{=x} \cdot \frac{d}{dx}\log(x) = x \frac{d}{dx}\log(x)$$

$F(x) = f(g(x))$

mit: $f(x) = e^x$

$g(x) = \log(x)$

$$\longrightarrow \boxed{\frac{d}{dx}\log(x) = \frac{1}{x}}$$

x^n für $n \in \mathbb{R}\setminus\{0\}$, $x > 0$:

$$\frac{d}{dx}(x^n) = \frac{d}{dx}\left(e^{\log(x)}\right)^n = \frac{d}{dx}\left(e^{n\log(x)}\right)$$

$$= e^{n\log(x)} \cdot \frac{d}{dx}\left(n\log(x)\right)$$

$$= x^n \cdot n \cdot \frac{1}{x}$$

$$= n x^{n-1}$$

Bsp: $\frac{d}{dx}\left(\frac{1}{\sqrt[3]{x}}\right) = \frac{d}{dx}\left(x^{-\frac{1}{3}}\right) = -\frac{1}{3}x^{-\frac{4}{3}}$

Wieso Quotientenregel unnötig?

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)\,g(x)^{-1}\right) = \frac{df}{dx}(x)\,g(x)^{-1} + f(x)\frac{d}{dx}\left(g(x)^{-1}\right)$$

Kettenregel

$$= \frac{\frac{df}{dx}(x)}{g(x)} + f(x)\left(-\frac{1}{g(x)^2}\,\frac{dg}{dx}(x)\right)$$

$$= \frac{\frac{df}{dx}(x)\, g(x) - f(x)\, \frac{dg}{dx}(x)}{g(x)^2}$$

$$F(x) = \frac{2x^2+1}{x^3-2} = (2x^2+1)(x^3-2)^{-1}$$

$$\longrightarrow \frac{dF}{dx}(x) = 4x\,(x^3-2)^{-1} + (2x^2+1)\left(-\frac{1}{(x^3-2)^2}\cdot 3x^2\right)$$

Aufgabe: Man leite ab:

(i) $\frac{1}{8}x^4 - \frac{1}{9}x^3 + \frac{1}{10}x^2 - \frac{1}{11} + \log(2x)$

Lösg.: $\frac{1}{2}x^3 - \frac{1}{3}x^2 + \frac{1}{5}x + \frac{1}{x}$

(ii) $(3x-2)(x^2+1)$ $\qquad$ $3(x^2+1) + (3x-2)2x$

(iii) $\frac{3x-2}{x^2+1}$ $\qquad$ $\frac{3}{x^2+1} - \frac{3x-2}{(x^2+1)^2}\, 2x$

(iv) $x^2 \sin(x)$ $\qquad$ $2x\sin(x) + x^2\cos(x)$

(v) $\frac{x^2}{\sin(x)}$ $\qquad$ $\frac{2x}{\sin(x)} - \frac{x^2}{\sin^2(x)}\cos(x)$

(vi) $(x-1)e^x$ $\qquad$ $+e^x + (x-1)e^x = e^x(+1+x-1) = (x\cancel{-x})e^x$

(vii) $\frac{x-1}{e^x}$ $\qquad$ $\frac{1}{e^x} - \frac{x-1}{e^{2x}}e^x = \frac{1-x+1}{e^x} = \frac{2-x}{e^x}$

Ableitung inverse Fkt.

$f, f^{-1}: \quad f(f^{-1}(x)) = x$

$$1 = \frac{d}{dx}x = \frac{d}{dx}\left(f(f^{-1}(x))\right)$$

$$= \frac{df}{dx}(f^{-1}(x)) \cdot \frac{df^{-1}}{dx}(x) \quad / : \frac{df}{dx}(f^{-1}(x))$$

$$\longrightarrow \boxed{\frac{df^{-1}}{dx}(x) = \frac{1}{\frac{df}{dx}(f^{-1}(x))}}$$

Bsp: $f(x) = e^x \longrightarrow \frac{df}{dx}(x) = e^x$

$f^{-1}(x) = \log(x)$

$$\longrightarrow \frac{d}{dx}\log(x) = \frac{1}{e^{\log(x)}} = \frac{1}{x}$$

Arcsin(x)

$f(x) = \sin(x)$ $\qquad$ Für $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$: $\arcsin(\sin(x)) = x$

$f^{-1}(x) = \arcsin(x)$

$$\longrightarrow \frac{d}{dx}\arcsin(x) = \frac{1}{\cos(\arcsin(x))}$$

Mit $\cos^2(x) + \sin^2(x) = 1 \longrightarrow \cos(x) = \sqrt{1 - \sin^2(x)}$

$+$, da $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\longrightarrow \cos(\arcsin(x)) = \sqrt{1 - \sin^2(\arcsin(x))}$

$= \sqrt{1 - x^2}$

$\sin^2(x) = (\sin(x))^2$

$$\longrightarrow \boxed{\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1 - x^2}}}$$

Arccos:

$$\boxed{\frac{d}{dx} \arccos(x) = \frac{1}{-\sin(\arccos(x))} = -\frac{1}{\sqrt{1 - x^2}}}$$

$\sin(x) = \sqrt{1 - \cos^2(x)}$