Ableitung: $$\frac{df}{dx}(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Bekannte Gesetze:

Konstante: $\frac{d}{dx} C = 0$

Potenz: $\frac{d}{dx} x^n = n x^{n-1}$ für $n \in \mathbb{N} \setminus \{0\}$

Summe: $\frac{d}{dx}\big(f(x) + g(x)\big) = \frac{df}{dx}(x) + \frac{dg}{dx}(x)$

Faktor: $\frac{d}{dx}\big(C f(x)\big) = C \frac{df}{dx}(x)$

Wurzel: $\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$

Trigo.: $\frac{d}{dx} \sin(x) = \cos(x)$; $\frac{d}{dx} \cos(x) = -\sin(x)$

Exp.: $\frac{d}{dx} e^x = e^x$.

Ableitung Produkt

Sei $p(x) = f(x) g(x)$

$$\longrightarrow \frac{dp}{dx}(x) = \lim_{h\to 0} \frac{p(x+h) - p(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) \overbrace{\boxed{- f(x+h)g(x) + f(x+h)g(x)}}^{0} - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \left(f(x+h) \frac{g(x+h) - g(x)}{h} + g(x) \frac{f(x+h) - f(x)}{h} \right)$$

$$= f(x) \cdot \frac{dg}{dx}(x) + g(x) \frac{df}{dx}(x).$$

I.e. **Produktregel:** (Leibnitzregel)

$$\boxed{\frac{d}{dx}\big(f(x) g(x)\big) = \frac{df}{dx}(x) g(x) + f(x) \frac{dg}{dx}(x)}$$

Bsp: $f(x) = (2x+1)(3x-2) \longrightarrow \frac{df}{dx}(x) = \left(\frac{d}{dx}(2x+1)\right)(3x-2) + (2x+1)\left(\frac{d}{dx}(3x-2)\right)$

$$= 2(3x-2) + (2x+1)3$$

$$= 6x - 4 + 6x + 3 = 12x - 1$$

(Kontrolle: $(2x+1)(3x-2) = 6x^2 - x - 2$

$\frac{d}{dx}(6x^2 - x - 2) = 12x - 1$ ✓)

Aufgabe: Man leite nach x ab: $\frac{d}{dx}\left(f(x)g(x)\right) = \frac{df}{dx}(x)g(x) + f(x)\frac{dg}{dx}(x)$.

$f(x) = \dots$	Lösg.: $\frac{df}{dx}(x) = \dots$
(i) $\sin(x)\cos(x)$	$\cos^2(x) - \sin^2(x)$
(ii) $e^x \sin(x)$	$e^x(\sin(x) + \cos(x))$
(iii) $\sqrt{x}\cos(x)$	$\frac{\cos(x)}{2\sqrt{x}} - \sqrt{x}\sin(x)$

(iv) Man leite eine Formel für die Ableitung von $f(x) = u(x)v(x)w(x)$ her.

$$\frac{df}{dx}(x) = \frac{du}{dx}(x)v(x)w(x) + u(x)\frac{d}{dx}\left(v(x)w(x)\right)$$

$$= \frac{du}{dx}(x)v(x)w(x) + u(x)\frac{dv}{dx}(x)w(x) + u(x)v(x)\frac{dw}{dx}(x)$$

zu (iii): $\frac{d}{dx}\left(\sqrt{x}\cos(x)\right) = \left(\frac{d}{dx}\sqrt{x}\right)\cos(x) + \sqrt{x}\left(\frac{d}{dx}\cos(x)\right)$

$$= \frac{1}{2\sqrt{x}}\cos(x) + \sqrt{x}\left(-\sin(x)\right) = \dots$$

Analog findet man: Quotientenregel:

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{df}{dx}(x)g(x) - f(x)\frac{dg}{dx}(x)}{g(x)^2}}$$

Bsp: $\frac{d}{dx}\left(\frac{x^2+1}{x^3-2x}\right) = \frac{\left(\frac{d}{dx}(x^2+1)\right)(x^3-2x) - (x^2+1)\left(\frac{d}{dx}(x^3-2x)\right)}{(x^3-2x)^2}$

$$= \frac{2x(x^3-2x) - (x^2+1)(3x^2-2)}{(x^3-2x)^2} = \dots = \frac{-x^4-5x^2+2}{(x^3-2x)^2}$$

Bsp: $\boxed{\frac{d}{dx}\tan(x)} = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\left(\frac{d}{dx}\sin(x)\right)\cos(x) - \sin(x)\left(\frac{d}{dx}\cos(x)\right)}{\cos^2(x)}$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \boxed{\frac{1}{\cos^2(x)}}$$

Potenzgesetz, negative Exponenten:

$\frac{d}{dx}(x^{-m}) = ?$ mit $m \in \mathbb{N}\setminus\{0\}$

Sei: $f(x) = 1$; $g(x) = x^m \longrightarrow x^{-m} = \frac{f(x)}{g(x)}$

$$\longrightarrow \frac{d}{dx}(x^{-m}) = \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\overset{0}{\overset{\parallel}{\frac{df}{dx}}}(x)g(x) - f(x)\frac{dg}{dx}(x)}{g(x)^2} = \frac{-mx^{m-1}}{x^{2m}} = -mx^{-m-1}$$

I.e. $\frac{d}{dx}x^{-m} = -mx^{-m-1}$ I.e. Potenzregel:

$$\boxed{\frac{d}{dx}x^n = nx^{n-1};\ n \in \mathbb{Z}\setminus\{0\}}$$

Bsp: $\frac{d}{dx}\left(\frac{1}{x^2}\right) = -\frac{2}{x^3}$; $\frac{d}{dx}\left(\frac{3}{x^7}\right) = -\frac{21}{x^8}$

$$\left[\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{d}{dx}(x^{-2}) = -2x^{-3} = -\frac{2}{x^3}\right]$$

Ableitung zusammengesetzte Fkt.: $F(x) = f(g(x))$

$$\frac{dF}{dx}(x) = \frac{d}{dx} f(g(x)) = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h\to 0}\left(\frac{f(g(x+h)) - f(g(x))}{\boxed{g(x+h) - g(x)}} \cdot \frac{\boxed{g(x+h) - g(x)}}{h}\right)$$

$$= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{H\to 0} \frac{f(g(x)+H) - f(g(x))}{H} \cdot \lim_{h\to 0}\frac{g(x+h) - g(x)}{h}$$

$H = g(x+h) - g(x)$

$$= \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x)$$

I.e.: **Kettenregel:**

$$\boxed{\frac{d}{dx} f(g(x)) = \underbrace{\frac{df}{dx}(g(x))}_{\text{heisst äussere Ableitung}} \underbrace{\frac{dg}{dx}(x)}_{\text{heisst innere Ableitung}}}$$

Bsp: (i) $F(x) = (2x+1)^3$

ist zusammengesetzt aus: $\left.\begin{array}{l} f(x) = x^3 \\ g(x) = 2x+1 \end{array}\right\} F(x) = f(g(x))$

$$\to \frac{dF}{dx}(x) = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x)$$

$$= 3(2x+1)^2 \cdot 2 = 6(2x+1)^2$$

⚠ $\left(\frac{dF}{dx}(x) \neq 3(2x+1)^2\right)$

$f(x) = x^3$

$\to \frac{df}{dx}(x) = 3x^2 \to \frac{df}{dx}(g(x)) = \frac{df}{dx}(2x+1) = 3(2x+1)^2$

[**Aufgabe:** $\frac{df}{dx}(x) = x^2 + 2x$; wie sieht $f(x)$ aus?]