Ableitung

Def.: $\frac{df}{dx}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Haben gesehen: $\boxed{\frac{d}{dx} C = 0}$ Konstante

$\boxed{\frac{d}{dx} x^n = n x^{n-1}}$ $n = 1, 2, 3, 4, \ldots$ Potenzregel

Polynome: Bsp: $f(x) = 7x^2 + 3x + 1$

Benötigen folgenden Satz über Grenzwerte:

Gegeben: $f(x)$, $g(x)$ mit $\lim_{x \to x_0} f(x)$; $\lim_{x \to x_0} g(x)$ ex.

$\Rightarrow$ (i) $\lim_{x \to x_0} \left(f(x) + g(x)\right) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$

(ii) $\lim_{x \to x_0} \left(f(x) \cdot g(x)\right) = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} g(x)$

(iii) $\lim_{x \to x_0} \left(\frac{f}{g}\right) = \frac{\lim f}{\lim g}$ (falls $\lim_{x \to x_0} g(x) \neq 0$).

Ableitung von Summen zweier Fkt.:

Sei $s(x) = f(x) + g(x)$

$$\longrightarrow \frac{ds}{dx}(x) = \lim_{h \to 0} \frac{s(x+h) - s(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \frac{df}{dx}(x) + \frac{dg}{dx}(x)$$

I.e. Summenregel: $\boxed{\frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx}}$

(Bsp: $g(x) = C$

$$\longrightarrow \frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx}\left(f(x) + C\right) = \frac{df}{dx}(x) + \overset{0}{\overset{\shortparallel}{\frac{dC}{dx}}} = \frac{df}{dx}(x)$$)

Sei $g(x) = C f(x)$

$$\longrightarrow \frac{dg}{dx}(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{C f(x+h) - C f(x)}{h}$$

$$= \lim_{h \to 0} C \cdot \frac{f(x+h) - f(x)}{h}$$

$$= C \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = C \frac{df}{dx}(x)$$

I.e. <u>Konstantenregel</u>: $\boxed{\frac{d}{dx} C f(x) = C \frac{df}{dx}(x)}$

<u>Ableitung von Polynomen</u>:

<u>Bsp</u>: $p(x) = 4x^3 + 3x \longrightarrow \frac{dp}{dx}(x) = \frac{d}{dx}\left(4x^3 + 3x\right)$

$$= \frac{d}{dx}(4x^3) + \frac{d}{dx}(3x)$$

$$= 4\frac{d}{dx}x^3 + 3\frac{d}{dx}x$$

$$= 4\cdot 3x^2 + 3\cdot 1$$

$$= \underline{12x^2 + 3}$$

<u>Allgemein</u>: $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$

$$\longrightarrow \frac{dp}{dx}(x) = n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \dots + 2a_2 x + a_1$$

(Oder: $p(x) = \sum_{k=0}^{n} a_k x^k \longrightarrow \frac{dp}{dx}(x) = \sum_{k=1}^{n} k a_k x^{k-1}$)

<u>Aufgabe</u>: Man leite ab: $f(x) = \dots$

(i) x^{13}

(ii) $2x^6$

(iii) 2^7

(iv) $x^2 + x^3$

(v) $3x^7 + 4x^2 + 2$

(vi) $C_1 x^{n_1} + C_2 x^{n_2}$
$C_1, C_2 \in \mathbb{R}$
$n_1, n_2 \in \mathbb{N} \setminus \{0\}$

<u>Lösg.</u>: $\frac{df}{dx}(x) = \dots$

$13x^{12}$

$12x^5$

0

$2x + 3x^2$

$21x^6 + 8x$

$C_1 n_1 x^{n_1 - 1} + C_2 n_2 x^{n_2 - 1}$

<u>Aufgabe</u>: $f(x) = 3x^4 - 2x^2 + 3x - 7$

Ges: Gleichung der Tang. an Graphen $y = f(x)$ bei $x_0 = \boxed{1}$.

<u>Lösg</u>: Pkt. auf Graph: $x_0 = 1 \longrightarrow y_0 = f(1) = 3 - 2 + 3 - 7 = \boxed{-3}$

i.e. $(x_0, y_0) = (1, -3)$.

Steigung Tang.: $\frac{df}{dx}(x) = 12x^3 - 4x + 3$

bei $x_0 = 1$: $\frac{df}{dx}(1) = 12 - 4 + 3 = \boxed{11}$

Geradengl.: $y = mx + b$ Bei $(x,y) = (x_0, y_0)$:

$-3 = 11 + b \longrightarrow b = -14$

$\longrightarrow \boxed{y = 11x - 14}$

Aufgabe: $f(x) = 3x^2 + 2x + 2$

Ges: Koord. Scheitelpkt.

Idee: Beim Scheitelpkt. gilt: $\frac{df}{dx} = 0$.

Lösg.: $\frac{df}{dx}(x) = 6x + 2 \stackrel{!}{=} 0 \longrightarrow x = -\frac{1}{3}$

$\longrightarrow y = 3\left(-\frac{1}{3}\right)^2 + 2\left(-\frac{1}{3}\right) + 2 = \frac{5}{3}$

I.e. $SP\left(-\frac{1}{3}, \frac{5}{3}\right)$

Wurzel: $f(x) = \sqrt{x}$

$$\frac{d}{dx}\sqrt{x} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{x+h} - \sqrt{x}\right)}{h} \cdot \frac{\left(\sqrt{x+h} + \sqrt{x}\right)}{\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{x + h - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

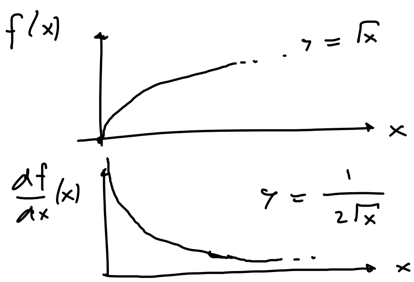
I.e. $\boxed{\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}}$

(Bem.: $\sqrt{x} = x^{\frac{1}{2}}$

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx} x^{\frac{1}{2}} = \frac{1}{2} x^{\frac{1}{2} - 1}$$

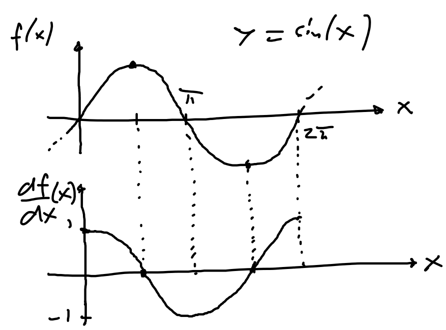
$$= \frac{1}{2} x^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x}}$$)



Sinus / Kosinus

$f(x) = \sin(x)$



$$\frac{df}{dx}(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$\underset{\text{Add.Thm.}}{=} \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x)\sin(h) + \sin(x)\left(\cos(h) - 1\right)}{h}$$

$$= \lim_{h\to 0} \Big(\underbrace{\frac{\sin(h)}{h}}_{\xrightarrow{h\to 0} 1} \cos(x) + \sin(x) \underbrace{\frac{\cos(h)-1}{h}}_{\xrightarrow{h\to 0} 0} \Big) = \cos(x)$$

I.e. $\boxed{\frac{d}{dx}\sin(x) = \cos(x)}$

Analog: $\boxed{\frac{d}{dx}\cos(x) = -\sin(x)}$

<u>Exp.-Fkt.:</u> Sei $f(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

$$\longrightarrow \frac{df}{dx}(x) = \frac{d}{dx}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right)$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

I.e.: $\boxed{\frac{d}{dx} e^x = e^x}$

<u>Aufgabe:</u>

Man leite nach x ab:

$$f(x) = 4x^7 + 2\sqrt{x} + 8(\sin(x) + \cos(x)) + 13e^x - 8$$

<u>Lösg.:</u> $\frac{df}{dx}(x) = 28x^6 + \frac{1}{\sqrt{x}} + 8(\cos(x) - \sin(x)) + 13e^x$

<u>Aufgabe:</u> Spannungsverlauf einer Lastabh. DC-Quelle:

$$U(I) = 16 - 3I^2$$

Wie gross muss der angeschlossene Widerstand sein für maximale Leistung?

<u>Lösg.</u> $P(I) = U(I)\cdot I = 16I - 3I^3$

$$\longrightarrow \frac{dP}{dI}(I) = 16 - 9I^2 \stackrel{!}{=} 0$$

$$\longrightarrow 16 = 9I_0^2$$

$$\longrightarrow I_0 = \sqrt{\frac{16}{9}} = \frac{4}{3}$$

$$\longrightarrow \boxed{R = \frac{U(I_0)}{I_0} = \frac{16 - 3I_0^2}{I_0} = \frac{\left(16 - 3\cdot\frac{16}{9}\right)3}{4} = \frac{32}{4} = 8}$$