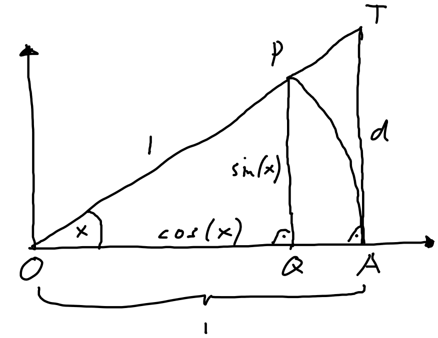
Zwei spezielle Grenzwerte

(i) $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x}$

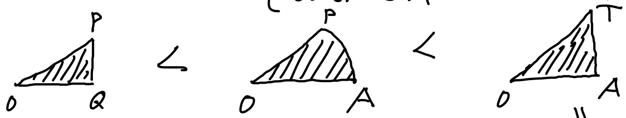


$x =$ Länge des Bogens

Ähnlichkeit: $\dfrac{\sin(x)}{d} = \dfrac{\cos(x)}{1} \longrightarrow \underline{d = \dfrac{\sin(x)}{\cos(x)}}$

(oder einfach: $\tan(x) = d$)

Flächen:

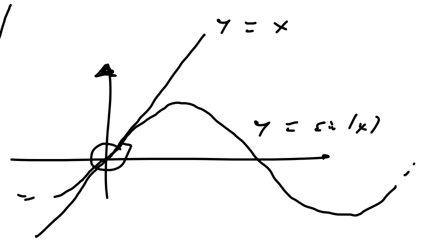


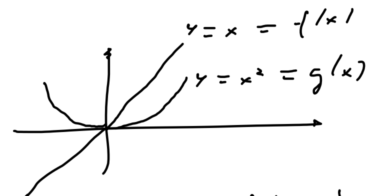
$$\frac{\cos(x)\sin(x)}{2} < \pi \frac{x}{2\pi} < \frac{d}{2}, \quad d = \frac{\sin(x)}{\cos(x)} \qquad \Big/ \cdot \frac{2}{\sin(x)}$$

$$\underline{\cos(x) < \frac{x}{\sin(x)} < \frac{1}{\cos(x)}} \qquad (*)$$

Aber: $\lim\limits_{x \to 0} \cos(x) = 1$; $\lim\limits_{x \to 0} \dfrac{1}{\cos(x)} = 1$

$\Rightarrow \lim\limits_{x \to 0} \dfrac{x}{\sin(x)} = 1 \Rightarrow \lim\limits_{x \to 0} \dfrac{1}{\frac{x}{\sin(x)}} = \boxed{\lim\limits_{x \to 0} \dfrac{\sin(x)}{x} = 1}$





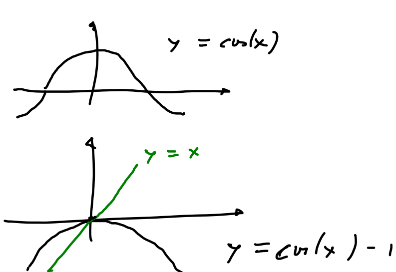
$\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)} \neq 1$

(ii) $\lim\limits_{x \to 0} \dfrac{\cos(x) - 1}{x}$

$$\boxed{\lim_{x \to 0} \frac{\cos(x) - 1}{x}} = \lim_{x \to 0} \frac{\cos(x) - 1}{x} \, \frac{\cos(x) + 1}{\cos(x) + 1}$$

$$= \lim_{x \to 0} \frac{\cos^2(x) - 1}{x(\cos(x) + 1)} = \lim_{x \to 0} \frac{-\sin^2(x)}{x(\cos(x) + 1)}$$

$$= \lim_{x \to 0} - \underbrace{\frac{\sin(x)}{x}}_{\xrightarrow{x \to 0} 1} \cdot \underbrace{\frac{\sin(x)}{\cos(x) + 1}}_{\xrightarrow{x \to 0} 0} = \boxed{0}$$



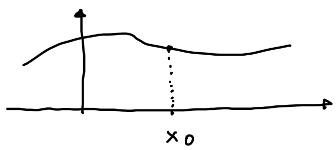
Stetigkeit

> **Def:** Sei $f(x)$ in einer Umgebung von x_0, inklusive x_0 definiert.
> $f(x)$ heisst <u>stetig an der Stelle x_0</u>, falls $\lim\limits_{x \to x_0} f(x)$ existiert und
> $$\lim_{x \to x_0} f(x) = f(x_0).$$
> Ansonsten heisst $f(x)$ unstetig bei x_0

<u>Bemerkungen:</u> (i) Stetigkeit bei x_0 setzt voraus dass $f(x_0)$ def. ist.

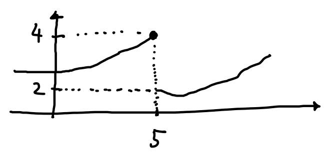
(ii) Stetigkeit entspricht einem Graphen, der ohne Absetzen des Stiftes gezeichnet werden kann.

<u>Bsp:</u> (i)



<u>f stetig bei x_0</u>

(ii)

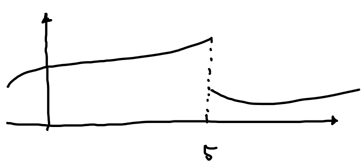


<u>f unstetig bei $x_0 = 5$</u>

$\lim\limits_{x \to 5} f(x)$ ex. nicht,

$$\lim_{x \to 5^-} f(x) = 4$$
$$\lim_{x \to 5^+} f(x) = 2$$

(iii)

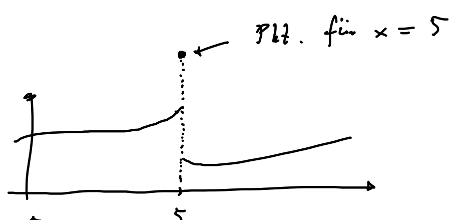


Sei $f(5)$ nicht def.

$\longrightarrow$ <u>f ist bei $x_0 = 5$ weder stetig noch unstetig</u>,

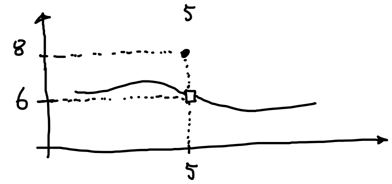
(f hat bei $x_0 = 5$ eine Def.-Lücke.)

(iv)



$\longrightarrow$ <u>unstetig</u>

(v)



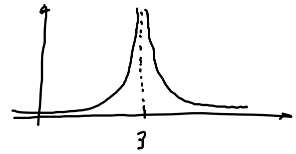
<u>unstetig</u> weil:

$$\lim_{x \to 5} f(x) \neq f(5)$$

$$\left(\lim_{x \to 5} f(x) = 6, \quad f(5) = 8 \right)$$

> **Def:** Eine Fkt. <u>$f(x)$ ist stetig</u>, wenn sie für alle x in ihrem Def.-Bereich stetig ist.

(vi)



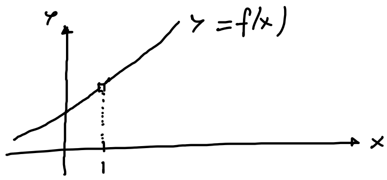
$f(x) = \frac{1}{(x-3)^2}$

f an Stelle $x_0 = 3$ weder stetig noch unstetig, da f bei $x_0 = 3$ nicht def.

Jedoch ist f stetig, da f an den Stellen wo f def. ist stetig ist.

Beheben einer Def. - Lücke

Sei $f(x) = \frac{x^2-1}{x-1} \longrightarrow$ Def. - Lücke bei $x = 1$.



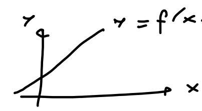
Es gilt: $f(1)$ ex. nicht

$$\left.\begin{array}{l} \lim\limits_{x \to 1^-} f(x) = 2 \\ \lim\limits_{x \to 1^+} f(x) = 2 \end{array}\right\} \lim\limits_{x \to 1} f(x) = 2$$

Man definiert:

$$\tilde{f}(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ 2 & x = 1 \end{cases}$$

Somit: $\tilde{f}(x) = x + 1$

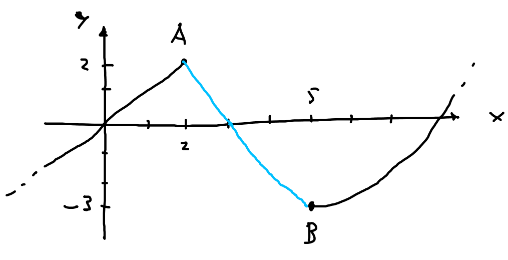


Aufgabe: Sei

$$f(x) = \begin{cases} x & : x \le 2 \\ ax + b & : 2 < x < 5 \\ (x-5)^2 - 3 & : x \ge 5 \end{cases}$$

Man finde a, b, so dass $f(x)$ stetig.

Lösg:



$A \to B$: $\Delta y = -5$, $\Delta x = 3$ $\longrightarrow$ $a = -\frac{5}{3}$

$y = ax + b = -\frac{5}{3}x + b$ (*)

A: $(2,2)$ in (*) $\longrightarrow$ $2 = -\frac{5}{3} \cdot 2 + b$

$\longrightarrow b = \frac{16}{3}$

$$\left(\text{I.e. } f(x) = \begin{cases} \dots \\ -\frac{5}{3}x + \frac{16}{3} \\ \dots \end{cases} \right)$$