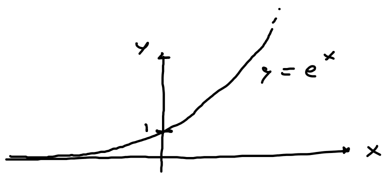
Natürliche Exponentialfkt.

Def: $$\exp(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

Graph:



Asymptote : $y = 0$ für $x \to -\infty$

Def.-Bereich : $\mathbb{R}$

Bild : $(0, \infty)$

Streng mon. wachsend

$\longrightarrow$ $\exp : \mathbb{R} \to (0, \infty)$, $x \mapsto y = e^x$ ist bijektiv.

Eigenschaft: $\underline{e^x e^y = e^{x+y}}$:

Herleitung : $$e^x e^y = \left(\sum_{m=0}^{\infty} \frac{x^m}{m!}\right)\left(\sum_{n=0}^{\infty} \frac{y^n}{n!}\right)$$
$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \cdots\right)$$

Tabelle :

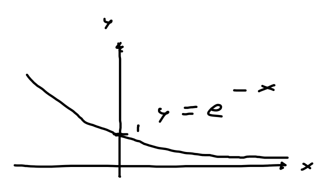
$e^y \downarrow$ / $e^x \rightarrow$	1	x	$\frac{x^2}{2}$	$\frac{x^3}{6}$	$\cdots$
1	1	x	$\frac{x^2}{2}$	$\frac{x^3}{6}$	
y	y	xy	$\frac{x^2}{2}y$	$\frac{x^3y}{6}$	
$\frac{y^2}{2}$	$\frac{y^2}{2}$	$x\frac{y^2}{2}$	$\frac{x^2y^2}{4}$	$\frac{x^3y^2}{12}$	
$\frac{y^3}{6}$	$\frac{y^3}{6}$	$x\frac{y^3}{6}$	$\frac{x^2y^3}{12}$	$\frac{x^3y^3}{36}$	
$\vdots$					

$$\longrightarrow e^x e^y = \underline{1} + \underline{x+y} + \underline{\tfrac{1}{2}(x^2 + 2xy + y^2)} + \underline{\tfrac{1}{6}(x^3 + 3x^2y + 3xy^2 + y^3)} + \cdots$$
$$= 1 + (x+y) + \frac{1}{2}(x+y)^2 + \frac{1}{6}(x+y)^3 + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!} = e^{x+y}.$$

I.e. $\boxed{e^x e^y = e^{x+y}}$

Wir sehen : $1 = e^0 = e^{x-x} = e^x e^{-x}$ / $: e^x$

$$\longrightarrow \underline{e^{-x} = \frac{1}{e^x}}$$



Natürlicher Logarithmus

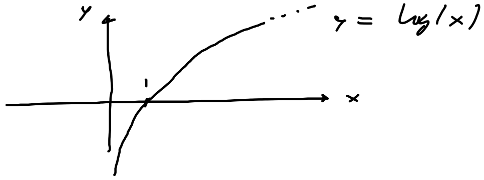
e^x ist bij. $\longrightarrow$ inverse Fkt. ex.

Wir definieren die Fkt. Natürlicher Logarithmus:

$\log : (0, \infty) \to \mathbb{R}$, $x \mapsto y = \log(x)$

als die inverse Fkt. zu $\exp(x)$. I.e. $\log(e^x) = x$.

Graph:



Bemerkung: In Literatur meist mit $\ln(x)$ bezeichnet.

Eigenschaften:

$$\text{(i)} \quad \log(xy) = \log(x) + \log(y)$$

$$\text{(ii)} \quad \log\left(\frac{x}{y}\right) = \log(x) - \log(y)$$

Herleitung von (i):

$$\text{Sei} \quad u = \log(x) \longrightarrow x = e^u$$

$$v = \log(y) \longrightarrow y = e^v$$

$$\longrightarrow \quad xy = e^u e^v = e^{u+v} \quad / \log(\ldots)$$

$$\log(xy) = u+v = \log(x) + \log(y) \qquad \square$$

Exponentialfkt. zur Basis $a > 0$:

Def: $\boxed{a^x = e^{x \log(a)}}$ $\longrightarrow$

$$\boxed{a^{x+y}} = e^{(x+y)\log(a)}$$

$$= e^{x\log(a) + y\log(a)}$$

$$= e^{x\log(a)} e^{y\log(a)}$$

$$= \boxed{a^x a^y}$$

Weitere Eigenschaft des log:

$$\log(x^y) = y \log(x)$$

Herleitung: $\quad x^y = e^{y\log(x)} \quad / \log(\ldots)$

$$\log(x^y) = y\log(x) \qquad \square$$

Potenzgesetze:

$$\text{(i)} \quad (a^x)^y = a^{xy}$$

$$\text{(ii)} \quad a^x b^x = (ab)^x$$

$$\text{(iii)} \quad \left(\frac{1}{a}\right)^x = a^{-x}$$

$$\text{(iv)} \quad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

Herleitung von (i):

$$(a^x)^y = \big(\underbrace{e^{x\log(a)}}_{a^x}\big)^y$$

$$= e^{y\log\left(e^{x\log(a)}\right)}$$

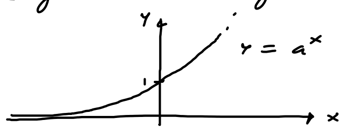
$$= e^{yx\log(a)} = a^{xy} \qquad \square$$

Graphen:

$a > 1$:

$$a^x = e^{x \overbrace{\log(a)}^{>0}}$$

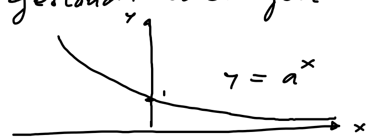
$\longrightarrow$ sieht aus "wie" e^x (gestaucht oder gestreckt ...)



$0 < a < 1$:

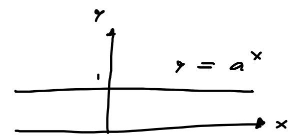
$$a^x = e^{x \overbrace{\log(a)}^{<0}}$$

$\longrightarrow$ sieht aus "wie" e^{-x} (gestaucht oder gestreckt)



$a = 1$:

$$a^x = e^{x \overbrace{\log(a)}^{=0}} = e^0 = 1$$



Logarithmus zur Basis $a \neq 1$:

Def: $\log_a : (0, \infty) \longrightarrow \mathbb{R}$

$x \longmapsto y = \log_a(x)$ ist die Umkehrfkt. von $y = a^x$.

(Es gilt: $\log_e(x) = \log(x)$)

Logarithmengesetze:

(i) $\log_a(xy) = \log_a(x) + \log_a(y)$

(ii) $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

(iii) $\log_a(x^y) = y \log_a(x)$

Basiswechsel:

$$\log_a(x) = \frac{\log(x)}{\log(a)}$$

Herleitung: Sei $u = \log_a(x)$

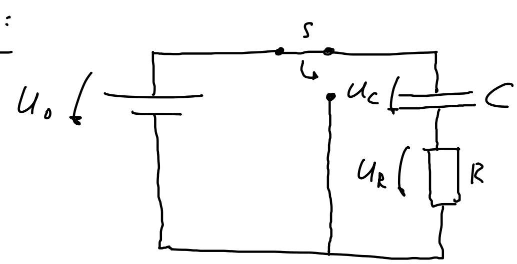
$\longrightarrow x = a^u \quad / \log(\ldots)$

$\log(x) = \log(a^u) = u \log(a) = \log_a(x) \log(a)$

$\longrightarrow \log_a(x) = \frac{\log(x)}{\log(a)} \quad \square$

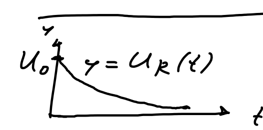
Mögliche Anwendungen:

(i) RC-Kreis:



Nach Umschalten:

$$U_R(t) = U_0 e^{-\frac{1}{RC} t}$$

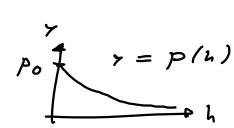


(ii) Barometrische Höhenformel:

$$p(h) = p_0 e^{-\frac{h}{7991 m}}$$

p: Luftdruck

h: Höhe über Boden



Grenzwert und Stetigkeit einer Funktion

Reelle Zahlenfolgen

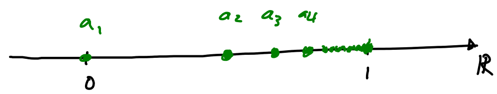
Bsp: $(a_n) = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$

Formel für das n-te Glied: $a_n = \frac{n}{n+1}$; $n = 0, 1, 2, 3, \dots$

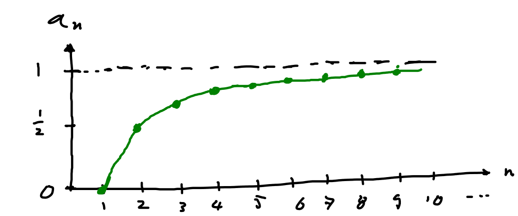
oder: $a_n = \frac{n-1}{n}$; $n = 1, 2, 3, 4, \dots$

Graphische Darstellung:

(i)



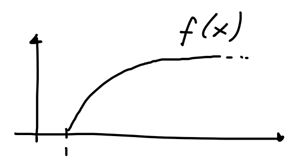
(ii)



Def: Eine Folge ist eine Funktion von $\mathbb{N}$ nach $\mathbb{R}$

$$a : \mathbb{N} \longrightarrow \mathbb{R}$$
$$n \longmapsto f(n) = a_n$$

(Mit $f(x)$ bekommen wir die "einhüllende Kurve"



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